At the morning session, four papers were presented:

1. "Analogues of De Moivre's theorem" by Professor John Tyler, U. S. Naval Academy.

2. "The theory of six numbers" by Professor Frank Morley, Johns Hopkins University, by invitation.

3. "On the conditions for the existence of a kinetic potential" by Professor J. H. Taylor, George Washington University.

4. "Probability in bombing as related to the angle of approach" by Professor C. C. Bramble, Postgraduate School, Naval Academy.

After lunch, all were guests of St. John's College; a remarkable exhibit had been arranged in the College Library by Professor G. A. Bingley, of books selected from the King William Collection of the College (the oldest extant of Colonial free public libraries: the Biblioteca Annapolitana of 400 volumes, presented in 1657), and from the private collections of President Gordon, Dr. R. T. H. Halsey, and Dr. Nathan Starr. The most noteworthy exhibits were:

Pacioli, Luca de Borgo: Divina Proportione; Venice 1509. (Plates attributed to Leonardo da Vinci.)

A second edition (1500) of Martiani Capella's famous encyclopaedia.

Numerous English works of the 18th century on building and architecture, one of them Wm. Pain's Practical House Carpenter, used by George Washington.

Maclaurin, Colin—Newton's Phil. Discourses 5th Ed. 1794.

Vol. 1 of Ozanam's Mathematical Course (5 vols.) London, 1712.

Tracts by Robert Boyle on Flame, Air and Explosives, London, 1672.

Freheri, D. Pauli—Theatri Vivorum Eruditione Clarorum, 2 vols. Nuremberg, 1688.

Two volumes of U. Aldrovanus, Bonn, 1667.

Andrew Wakely's Mariner's Compass and Tables, London, 1758.

The afternoon session completed the program with two papers:

5. "Magic squares" by H. M. Robert, U. S. Naval Academy, by invitation.

6. "Tritangent circles of the rational bi-cubic" by W. K. Morrill, Johns Hopkins University.

Abstracts of some of the papers follow:

2. Professor Morley spoke as follows: Ordinary algebra is concerned with unrestricted numbers, usually called complex. These numbers are represented by points of a plane, or by points of a sphere Ω, or by points of certain other rational surfaces. When we consider pairs of numbers, that is quadratics $q \equiv \alpha x^2 + \beta x + \gamma$, the sphere is indicated. It is appropriate to call it the Celestial Sphere, and to regard it as the infinity of space around one. To two points x, y of Ω we attach the arc of a circle orthogonal to Ω, with the end points x, y. Thus an arc represents a quadratic q, and q names an arc.

Two quadratics q_1 and q_2 have the bilinear invariant $q_{12} = q_{21}$. When it vanishes, the arcs cut at right angles; let us say that they are normal. Two

quadratics have also the Jacobian $j_{12} = -j_{21}$. This is represented by the common normal of the two arcs. The form j_{12}/q_{12} is an absolute covariant of q_1 and q_2 (in this order). For any number of quadratics q_i, when ordered, the quadratic

$$j_{12}/q_{12} + j_{23}/q_{23} + \cdots + j_{n1}/q_{n1} \tag{1}$$

is of interest.

Taking six quadratics, $q_i = \alpha_i x^2 + \beta_i x + \gamma_i$, subject to the relation

$$\left| \alpha_i^2 \beta_i^2 \gamma_i^2 \beta_i \gamma_i \gamma_i \alpha_i \alpha_i \beta_i \right| = 0, \tag{2}$$

or in other words taking six pairs of numbers which are in a biquadratic involution (symmetric two-to-two correspondence) we have in the unrestricted domain the results of the "hexagramma mysticum" in the unrestricted domain[1] where $\alpha_i : \beta_i : \gamma_i$ are real. Here (1), with $n = 6$, takes the place of a Kirkman Point.

If we change the ordering by cyclic permutation of 2, 4, 6, we have the three Kirkman arcs

$$j_{12}/q_{12} + j_{23}/q_{23} + j_{34}/q_{34} + j_{45}/q_{45} + j_{56}/q_{56} + j_{61}/q_{61}$$

$$j_{14}/q_{14} + j_{43}/q_{43} + j_{36}/q_{36} + j_{65}/q_{65} + j_{52}/q_{52} + j_{21}/q_{21}$$

$$j_{16}/q_{16} + j_{63}/q_{63} + j_{32}/q_{32} + j_{25}/q_{25} + j_{54}/q_{54} + j_{41}/q_{41}$$

whose form is 0. The three such arcs have then a common normal, apart from the relation (2).

In general a theorem of planar projective geometry, when stated with reference to a conic, passes into a theorem of arcs by omitting the restriction of real numbers. And by expanding the universe so that the radius of Ω becomes ∞, we have a theorem on the lines of a Euclidean space. Thus the lines of a Euclidean space are a representation of the theory of quadratics (under homographics).

4. Professor Bramble said that under the assumptions of normal dispersion and no correlation of deviations in range and deflection, the probability of dropping a bomb on a rectangular target is written as a function of the angle of approach. This probability is a maximum for approach along the longer axis of the rectangle if the ratio of the longer axis to the angle of dispersion in range is not less than the ratio of the shorter axis to the average dispersion in deflection. In any case it is shown that the maximum probability occurs within a certain angle.

5. Professor Robert gave a brief account of the history of magic squares, their classification, methods of construction and illustrations of various types.

PAUL CAPRON, *Secretary*

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[1] Salmon, Conic Sections.

THE INDIAN ORIGIN OF THE MODERN PLACE-VALUE ARITHMETICAL NOTATION

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

In the *Bakhshali Manuscript* (1927) edited by G. R. Kaye we read that "there is no sound evidence of the employment in India of a place-value system earlier than about the ninth century A.D." (page 82). This view has been so insistently given prominence in his articles published in various journals of India and Europe ever since July, 1907, that European and American scholars who have to depend for their knowledge of Indian mathematics on second-hand sources (i.e. on translations and criticisms of Indian mathematical and astronomical works in European languages) have had no other alternative than to accept the view. Accordingly we find the late Professor Cajori concluding his article[1] on the origin of the numerals as follows:

"As a by-product of the discussion of recent years we must admit that, on the evidence presented, the claim that our numerals and the zero were used in India as early as the fifth century must be abandoned, our earliest apparently reliable evidence belongs to the ninth century. . . .These corrections are due G. R. Kaye."

In this paper it is proposed to show that the above view of Kaye, accepted by Cajori and others, is not warranted by facts.

In the October (1927) issue of this MONTHLY I showed that an enunciation of the modern place-value arithmetical notation occurs for the first time in the *Āryabhaṭīyam* which was written by the elder Ārabhaṭa in the year 499 A.D. Indian astrological, astronomical, and philosophical works[2] written after this date contain indisputable evidence in abundance to show that the modern notation has since that date been employed in India. As the use of the notation in India from the ninth century onwards is no longer disputed, I shall here discuss the evidence available in the following Indian works which are admitted by all authorities to have been written before the ninth century A.D.:

(1) Jīva Śarmā's astrological work[3] (written before 550 A.D.).
(2) Varāhamihira's *Vṛhatsaṃhitā* (c. 550 A.D.).
(3) Varāhamihira's *Pañca-siddhāntikā* (6th century A.D.).
(4) Brahmagupta's *Brāhma-spuhṭasiddhānta* (628 A.D.).
(5) Brahmagupta's *Khaṇḍakhādyaka* (665 A.D.).
(6) Lalla's *Śiṣyadhīvṛddhida* (7th century A.D.).[4]

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[1] Scientific Monthly, Vol. 9, No. 5, November, 1919.

[2] I have excluded Subandhu's *Vāsavadattā*—a literary work of about 620 A.D. in which stars are compared to ciphers (*Śūnya-vindavaḥ*); for, I hold that the cipher came into use long before the invention of the place-value arithmetical notation.

[3] Jīva Śarmā is anterior to Varāhamihira as he is mentioned as an authority by the latter (*Vrhajjātaka*, VII, 9).

[4] Sudhākar Dvivedī in his *Gaṇaka-taraṅgiṇī* holds that Lalla preceded Varāhamihira. J. C. Ray in his *Āmāder Jyotiṣī O Jyotiṣ* takes Lalla to be a contemporary of Brahmagupta. A. B. Keith

(7) A commentary (*bhāṣya*) on Patañjali's *Yogasūtra* by Vyāsa (not later than the 7th century A.D.).[1]

(8) A commentary on *Brahma Sūtra* by Saṅkara (8th century)[2]

A quotation from Jīva Śarmā occurs in Bhattotpala's commentary on Verse 9 of Chapter VII of Varāhamihira's *Vṛhajjātaka* and contains two examples (viz., *dvi-yama* (22) and *vedāgni* [=*veda*+*agni*] (34)) of a method of stating, in terms of word-numerals,[3] numbers expressed in the modern place-value notation.

Two such number-expressions *nava*+*aṣṭa-pañca*+*aṣṭa*[4] (8589) and *śūnya-śara*+*aga-rāma* (3750) occur in the *Vṛhatsaṃhitā* (Chapter VIII, Verse 20). But similar number-expressions occur in great abundance in the works (3)—(6) mentioned above. Bühler holds that they "presuppose the existence of the decimal notation."[5] They also led Burnell to conclude, "It is thus perfectly clear that the Indians knew of numerals with a value according to position in the sixth century A.D."[6] As this opinion of these two authorities has been rejected by historians of mathematics under the influence of Kaye's writings, it is necessary to undertake a critical examination of these number-expressions in order to find the underlying notation.

Although it must be admitted that the figures found in any manuscript copy of a work can not always be taken faithfully to represent the actual forms of like figures of a distant earlier period, the same can not be said of any number stated not in figures but in words forming part of a Sanskrit verse. For example, let us take the expression *śūnya-śara*+*aga-rāma* quoted above. It stands for 3750. Either this expression must be copied as such or the component words must be so replaced by their equivalents that the metre may not be interfered with. The word *śūnya* which means 0 can not be changed into its equivalent *kha* or *ambara* as they have not the same number of syllables; but the word *pūrṇa* can be used for it as (i) *śūnya* and *pūrṇa* have the same number of syllables and (ii) the long syllable is followed by the short in both. It is difficult to replace the word *śara* (which means 5) without violating the metre. The word *aga* can be replaced by *adri*, *ṛṣi*, or *aśva*. The word *rāma* can

places him after Brahmagupta. For the sake of argument only I have here accepted the latest dates assigned to Lalla, Vyāsa, and Śaṅkara so that there may not arise any charge of antedating these authorities.

[1] S. N. Dās Gupta takes 400 A.D. to be the date of this commentary (*History of Indian Philosophy*, Vol. I, p. 212). J. H. Wood appears to assign it to the 6th century while A. B. Keith writes, that Vyāsa's "date is probably before Māgha" whom he places about 700 A.D. (*History of Sanskrit Literature*, pp. 490, 124).

[2] *Encyclopædia of Religion and Ethics*, Vol. II, p. 185.

[3] By a word-numeral I mean a word denoting a number not necessarily less than ten.

[4] In this and other number-expressions the word-numerals have been separated by the sign − or +, the latter sign indicating that the vowels separated by it have been joined in the text according to the rules for the conjunction of vowels.

[5] *Indian Antiquary*, Vol. XXXIII, 1904, Appendix, p. 861.

[6] *South Indian Palæography*, p. 62.

be changed into *kāla* or *loka*. Hence the expression *śūnya-śara+aga-rāma* can be altered into *pūrṇa-śara+adri-kāla* or *pūrṇa-śara+aśva-loka*. But in any case the notation remains the same. In dictating the number fifty expressed in the modern notation we may say "five zero" or "five naught." By replacing the word "zero" by the word "naught" we do not alter the notation. There is thus no reason to doubt that the notation behind the number-expressions which occur in the works of Jīva Śarmā and subsequent Indian authors referred to above existed as such at the times when these works were written.

What is this notation? In Varāhamihira's *Pañca-siddhāntikā* edited by Thibaut and Dvivedī we find four number-expressions, namely, *aṣṭa-yama-pakṣa* (228), *manu-śara* (514), *guṇa-kha-sapta* (703), and *aṣṭa-nava-rasa* (698) on the second page of the text. In the first, third and fourth of these expressions words have been used which denote numbers less than ten. In these three expressions we recognize the existence of the modern place-value notation with ten word-numerals including the zero. In such cases each word-numeral serves the purpose of a modern digit. But, when we come across number-expressions like *manu-śara* (514), *rada-veda*[1] (432), *rasa-tithi-yamala* (2156), and *danta+aṣṭa-yama* (2832) which consist of less word-numerals than the number of digits required to express the corresponding numbers in our modern notation, we are, at first sight, led to doubt the existence of any place-value notation. Nay, we may go farther and assert that there does not seem to be any sound principle behind these number-expressions or that the above two classes of number-expressions have originated from two different sources so that their occurrence in the same work is conclusive proof of its spuriousness.

The difficulty of reconciling the above-mentioned two classes of number-expressions vanishes when it is recognized that the expressions give the corresponding numbers expressed in our modern place-value notation by stating the digits (either one or two at a time to suit the metre) beginning with the units.[2] The practice of stating the digits of a number to save the time and labour of enumerating it fully is not now uncommon in the West, as is seen in reading telephone numbers. India also followed—and still follows—the same practice in metrical composition involving numbers, with this difference—that India generally[3] was more rational and scientific in this case than the West now is. For, when one begins with the digit in the units' place and enumerates the other digits in succession, the local value of each digit is apparent; whereas, when one begins with the digit in the highest place, the hearer is quite in the dark as

[1] This and the following two expressions are taken from *Brāhma-sphuta-siddhānta* (I, 15; II, 3 and 4).

[2] This view has been taken also by Burgess who writes that "the method of writing numbers, large or small, is by naming the figures which compose them" (Introductory Note to his Translation of the *Sūrya Siddhānta*, p. 143).

[3] In his expressions for numbers in terms of letter-numerals Junior Āryabhata (10th century) states the digits beginning with the highest place (*Mahāsiddhānta*, pp. 4, 5, etc.). When he uses word-numerals, he begins with the units' place according to the existing convention (Ibid, Ch. XV).

to the local values of the digits until all the digits are exhausted.[1] As the year one thousand nine hundred and thirty-two is conveniently stated as "nineteen thirty-two," so the number 514 can be conveniently stated as "five fourteen" or "fifty-one four" according to Western practice or as "fourteen (*manu*) five (*śara*)," "four fifty-one" according to Indian practice. Hence the expression *manu-śara* for 514. Similarly we get the other expressions involving less word-numerals than the digits in the corresponding numbers. The expressions *śūnyāni sapta* (seven zeros) *rada-vedāḥ* (432) for the number 4320000000 and *khacatuṣṭaya-yama-śara-guṇa-śaśi-tri-veda*[2] (four zeros, two, five, three, one, three, four) for the number 4313520000 dispels all doubt as to the principle underlying all the number-expressions in terms of word-numerals which occur in the works mentioned above.

That the modern arithmetical notation is the basis of all the number-expressions which occur in the Indian works from the sixth century onwards will be clear from the following different ways[3] in which the number 123249 can be expressed with word-numerals in accordance with the demands of the metre used. The number can be stated thus:

(i) With six word-numerals

as *nanda-veda-netra-guṇa-pakṣa-śaśī* (1,2,3,2,4,9)
9 4 2 3 2 1

(ii) With five word-numerals

as (a) *tāna-netra-guṇa-pakṣa-śaśī* (1,2,3,2,49)
49 2 3 2 1

or as (b) *nanda-jina-guṇga-pakṣa-śaśī* (1,2,3,24,9)
9 24 3 2 1

or as (c) *nanda-veda-danta-pakṣa-śaśī* (1,2,32,4,9)
9 4 32 2 1

or as (d) *nanda-veda-netra-vikṛti-śaśī* (1,23,2,4,9)
9 4 2 23 1

or as (e) *nanda-veda-netra-guṇa-māsa* (12,3,2,4,9)
9 4 2 3 12

(iii) With four word-numerals

as (a) *tāna-danta-takṣa-śaśī* (1,2,32,49)
49 32 2 1

or as (b) *tāna-netra-vikṛti-śaśī* (1,23,2,49)
49 2 23 1

or as (c) *tāna-netra-guṇa-māsa* (12,3,2,49)
49 2 3 12

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[1] When a number is dictated to a copyist with a left-to-right script, the Western practice is more advantageous than the Indian.

[2] These two expressions are taken from *Brāhma-sphuta siddhānta*, Chapter I, Verses 15 and 52.

[3] The digits are stated from right to left, one or two at a time.

or as (d) *nanda-jina-vikṛti-śaśi* (1,23,24,9)
9 24 23 1

or as (e) *nanda-jina-guṇa-māsa* (12,3,24,9)
9 24 3 12

or as (f) *nanda-veda-danta-māsa* (12,32,4,9)
9 4 32 12

(iv) With three word-numerals
as *tāna-danta-māsa* (12,32,49)
49 32 12

As it is the previous knowledge of the modern notation which has made it possible in the West to state the number "one thousand nine hundred and thirty-two" as "one nine three two" or as "nineteen thirty-two," so it is the previous existence in India of the modern place-value arithmetical notation, which has made it possible for the Indian authors from the sixth century onwards to express the same number differently with suitable word-numerals as shown above.

In the February (1929) issue of the *Isis* I have shown that Vyāsa's commentary on Patañjali's *Yogasūtra* and Śaṅkara's commentary on *Brahma Sūtra* also point to the use of the modern place-value notation in India at least as early as the sixth century, A.D. For the convenience of my readers the following extracts are taken from the journal (page 135):

The following passage occurs in the *Vyāsa Bhāṣya* (*Bibhūtipāda*, 13th *sūtra*) of the *Yogasūtra* of Patañjali:

"Yathaikā rekhā śatasthāne śataṃ daśathāne daśa ekañcaikasthāne"[1] etc.

It may translated thus:

"Just as the same stroke (or figure) represents a hundred, a ten, or a unit according as it is in the hundred's place, the ten's place, or the unit's place," etc.

Professor J. H. Wood in this connection observes[2] that contrary to G. R. Kaye's opinion "the place system of decimals was known as early as the sixth century A.D."

Śaṅkara's (c. 750 commentary on *Brahmasūtra* contains the following passage:

"Yathā caikāpi satī rekhā sthānānyatvena niveśyamānaikadaśaśatasahasrādipratyayabhedamanubhavati" etc. (*Śārīrikabhāṣya*, 2.2.17).

It may be translated thus:

"Just as the same stroke (or figure) conveys different ideas such as a unit, a ten, a hundred, a thousand, etc., according to the place in which it is set down," etc.

"The use of such passages for the purpose of elucidation of abstruse philosophical doctrines shows that the place-value decimal notation was extremely popular at the time when the above commentaries were written."

[1] My attention was kindly drawn to this passage and the next one by B. B. Datta.

[2] *English Translation of the Yoga System* of Patañjali (HOS), p. 216, foot-note.

Taking into consideration the absence of facilities of communication in those early times, it would not be extravagant to suppose that the place-value decimal system of notation was at least two centuries old at the time of Vyāsa.[1]

It will thus be seen that there is no reason to be surprised at Severus Sebokht's (7th century) highly appreciative reference to the Indian method of computation by means of nine signs (meaning thereby the employment of the place-value decimal notation),[2] the presence of the Indian figures in Syria in 662 A.D.[3] and the fact that the Arab writers from the early ninth century on acknowledged the Indian origin of the modern place-value notation.[4]

COMPARISON OF CERTAIN PROBABILITIES

By WILLIAM DOWELL BATEN, University of Michigan

Probabilities of a continuous variable may be considered as areas under a probability curve between certain limits. The entire area under the curve for the interval of definition is unity, which represents the sum of all possible probabilities of the variable under consideration. The probability that the variable x lies on the interval (a, b) is $\int_a^b f(x)dx$, where $f(x)dx$ is the probability that x lies on the interval $(x, x+dx)$.

When the probability function or probability law for the variable x is known, the probability laws for functions of x can be determined.[5] In some cases it is necessary to compare probabilities of various functions of x. These comparisons are obtained by considering two areas under the respective probability laws on an interval of comparison. Generally this interval of comparison is a small interval which includes the abscissa of the greatest value of the law. In error laws the interval which is of most importance is that which includes the probable value, or the mean.

In the following theorems several intervals of comparison will be considered. Frequently the theorems are only true in the interval mentioned.

THEOREM 1. *A necessary and sufficient condition that the probability of bx is greater than the probability of x, is that the positive constant b is less than 1; provided the interval of comparison is $(-c, d)$ and that x extends over the interval $(-k, h)$, where $k>c$ and $h>d$.*

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[1] If the date (400 A.D.) assigned to Vyāsa by S. N. Dās Gupta be correct, the date of invention of the modern notation in India would be pushed back several centuries earlier. But in that case it would be difficult to explain the presence of the enunciation of the notation in the *Āryabhaṭiyam*.

[2] D. E. Smith, *History of Mathematics*, Vol. II (1925), pp. 64 and 65.

[3] Ibid, p. 72. Also Kaye, *Indian Mathematics*, p. 31.

[4] Smith and Karpinski, *The Hindu-Arabic Numerals*, pp. 4 and 10. Also Cajori, *A History of Mathematics*, (1922), pp. 102 and 103.

[5] See Dodd: *The frequency law of a function of one variable*, Bulletin of the American Mathematical Society, Vol. 31 (1925), pp. 27–31.

PROOF: Let $f(x)dx$ be the probability that the variable x lies on the interval $(x, x+dx)$, then the probability that, $y=bx$, lies on the interval $(y, y+dy)$ is[1]

$$F(y)dy = (1/b)f(y/b)dy.$$

If the probability that y lies on the interval $(-c, d)$ is greater than the probability that x lies on this interval, then

$$(1/b)\int_{-c}^{d} f(y/b)dy > \int_{-c}^{d} f(x)dx,$$

or

$$\int_{-c/b}^{d/b} f(z)dz > \int_{-c}^{d} f(x)dx.$$

If this is true the interval $(-c/b, d/b)$ is greater than the interval $(-c, d)$, or

$$(1/b)(d+c) > (d+c),$$

or

$$b < 1.$$

The sufficiency of the condition can be proved by reversing the steps of the foregoing proof.

It is readily seen that the quantity c must be non-negative.

The quantity k may be zero, in which case c is zero. If the maximum of the law is not at the origin, the origin can be changed to the maximum or greatest value of the probability function.

If each variable x_i $(i=1, 2, 3, \cdots, n)$ is subject to the same probability law defined on the interval $(-k, h)$, then the law for the sum of n variables can be found.[2] Let $y=(1/n)\sum_{i=1}^{n} x_i$, then according to the above theorem the probability of the mean for the interval $(-c, d)$ is greater than the probability for the sum, since here $b=1/n$, and is less than unity when n is greater than 1. The probability law for the mean may be zero in part of $(-c, d)$.

THEOREM 2. *Let the probability law for the variable x be $f(x)$, which is defined on the interval $(-c, a)$, where c and a are positive quantities. If the positive quantity b is less than* 1, *there exists an interval of comparison $(0, d)$ in which the probability of bx^k is greater than the probability of x^s, provided $k>s>1$.*

PROOF: If the law for x is $f(x)$, then the law for $y=bx^k$ is[3]

$$F(y) = \frac{p \cdot y^{(1-k)/k}}{kb^{1/k}} \cdot f\{(y/b)^{1/k}\},$$

and the law for $z=x^s$ is

$$\Psi(z) = \frac{q \cdot z^{(1-s)/s}}{s} \cdot f\{(z)^{1/s}\},$$

[1] Dodd, loc. cit., Bulletin.

[2] Dodd, *The frequency law of a function of variables with given frequency laws*, Annals of Mathematics, Ser. 2, vol. 27 (1925) pp. 12–20.

[3] Dodd, loc. cit., Bulletin.

where p and q are 2 if k and s are even and 1 if k and s are odd. The probability that bx^k lies on the interval $(0, d)$ is greater than the probability that x^s lies on this interval if

$$\int_0^d F(y)dy > \int_0^d \Psi(z)dz, \text{ or if}$$

$$\frac{p}{kb^{1/k}} \int_0^d y^{(1-k)/k} f\{(y/b)^{1/k}\} dy > \int_0^d (q/s) z^{(1-s)/s} f\{(z)^{1/s}\} dz,$$

or if

$$\int_0^{(d/b)^{1/k}} f(t)dt > \int_0^{d^{1/s}} f(v)dv.$$

Let $s = rk$, where $r < 1$. The last inequality is true if

$$(d/b)^{1/k} > (d)^{1/rk},$$

or if

$$\log d < \{r/(r-1)\} \log b.$$

The quantity $r/(r-1)$ is negative and the logarithm of b is negative; hence the logarithm of d is less than some positive number, which determines a limit for the interval of comparison $(0, d)$.

The theorem is true if $s = k$ as can be seen from

$$(d/b)^{1/k} > (d)^{1/k}.$$

The theorem is also true if $s = 1$.

THEOREM 3. *If* $\sum_{i=1}^n b_i < \sum_{i=1}^m k_i$, *then the probability of* $\sum_{i=1}^n b_i x_i$ *is greater than the probability of* $\sum_{i=1}^m k_i x_i$, *provided the probability law for each variable is*

$$f(x_i) = \frac{h}{\pi(h^2 + x_i^2)},$$

and the interval of comparison is $(-c, d)$.

PROOF: The probability function for $y = b_i x_i$, is

$$F(y) = \frac{b_i h}{\pi} \cdot \frac{1}{b_i^2 h^2 + y^2},$$

the probability function for $\sum_{i=1}^m k_i x_i = u$, is

$$\Psi(u) = \frac{\sum_{i=1}^m k_i h}{\pi} \cdot \frac{1}{h^2 \left(\sum_{i=1}^m k_i \right)^2 + u^2},$$

the probability function for $w=\sum_{i=1}^{n} b_i x_i$, is

$$\theta(w) = \frac{\sum_{i=1}^{n} b_i x_i}{\pi\left(h^2\left(\sum_{i=1}^{n} b_i\right)^2 + w^2\right)}.$$

The probability that the sum $\sum_{i=1}^{n} b_i x_i$ lies in the interval $(-c,d)$ is greater than the probability that the sum $\sum_{i=1}^{m} k_i x_i$ lies in this interval if

$$\frac{h\sum_{i=1}^{n} b_i}{\pi}\int_{-c}^{d}\frac{du}{h^2\left(\sum_{i=1}^{n} b_i\right)^2 + u^2} > \frac{h\sum_{i=1}^{m} k_i}{\pi}\int_{-c}^{d}\frac{dw}{h^2\left(\sum_{i=1}^{m} k_i\right)^2 + w^2},$$

and this is true since $\sum_{i=1}^{n} b_i < \sum_{i=1}^{m} k_i$.

Nothing was stated about each b_i and each k_i being less than 1. In fact the theorem is true if some are greater than 1 and some less than 1, as long as the sum of the b's is less than the sum of the k's. If $k_i=1$ then the probability of $\sum_{i=1}^{n} b_i x_i$ is greater, for the interval $(-c, d)$, than the probability of $\sum_{i=1}^{n} x_i$, if $\sum_{i=1}^{n} b_i < n$.

THEOREM 4. *If the probability function for the variable x_i, $(i=1, 2, \cdots, n)$, is the Gaussian function, then the probability that*

$$-c \leqq \sum_{i=1}^{n} b_i x_i \leqq d,$$

is greater than the probability that

$$-c \leqq \sum_{i=1}^{m} k_i x_i \leqq d,$$

provided

$$\sum_{i=1}^{n} b_i^2 < \sum_{i=1}^{m} k_i^2.$$

PROOF: If the law for x_i is $h/\sqrt{\pi}\cdot e^{-h^2 x_i^2}$, then according to a method given by Dodd[1] the laws for the sums $\sum_{i=1}^{n} b_i x_i = u$ and $\sum_{i=1}^{m} k_i x_i = w$ are respectively

$$F(u) = \frac{h}{(\pi)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}} e^{-h^2 u^2 / \sum_{i=1}^{n} b_i^2},$$

$$\Psi(w) = \frac{h}{(\pi)^{1/2}\left(\sum_{i=1}^{m} k_i^2\right)^{1/2}} e^{-h^2 w^2 / \sum_{i=1}^{m} k_i^2}.$$

[1] Dodd, loc. cit., Annals.

The theorem is true if

$$\int_{-c}^{d} F(u)du > \int_{-c}^{d} \Psi(w)dw,$$

which is true if

$$\sum_{i=1}^{n} b_i^2 < \sum_{i=1}^{m} k_i^2 .$$

This shows that the probability of the sum $\sum_{i=1}^{m} x_i$ is greater than the probability of the sum $\sum_{i=1}^{n} x_i$, on the interval $(-c, d)$, if $m<n$.

In a weighted mean the sum $\sum b_i = 1$, hence the arithmetic mean has the greater probability for the interval mentioned. If however the $\sum_{i=1}^{n} b_i^2$ is less than $1/n$, then this kind of weighted mean has greater probability than the probability of the mean.

If the law for the individual variable x_i is an even function the probability for the sum $\sum_{i=1}^{n} b_i x_i$ can be compared with the probability for the sum $\sum_{i=1}^{m} k_i x_i$ by examining two integrals of the types[1]

$$p_{nb}(u) = (1/\pi) \int_0^\infty f(b_1 t) f(b_2 t) \cdots f(b_n t) dt,$$

$$p_{mk}(u) = (1/\pi) \int_0^\infty f(k_1 t) f(k_2 t) \cdots f(k_m t) dt.$$

The nature of the functions under the integral signs will have to be examined in each particular case before it can be stated that the probability of the first is greater than the probability of the second for some interval of comparison.

THEOREM 5. *Under the Gaussian Law of error the probable value of any given power of the absolute error of one weighted mean with weights* $p_1, p_2, \cdots, p_n$, *is less than that of a second weighted mean with weights* $q_1, q_2, \cdots, q_m$, *if* $\sum_{i=1}^{n} p_i^2 < \sum_{j=1}^{m} q_j^2$.

PROOF: If the law for the individual variables is $he^{-h^2x^2}/\sqrt{\pi}$, then the law for the first weighted mean is

$$P_n(u) = \frac{P}{(\pi)^{1/2}} e^{-P^2u^2}, \text{ where } P^2 = h^2 / \sum_{i=1}^{n} p_i^2 ;$$

and the law for the second weighted mean is

$$Q_m(w) = \frac{Q}{(\pi)^{1/2}} e^{-Q^2w^2}, \text{ where } Q^2 = h^2 / \sum_{i=1}^{m} q_i^2 .$$

[1] Dodd, loc. cit., Annals.

The probable value of the absolute error of the first raised to the power k is less than the absolute error of the second raised to this same power if

$$\int_{-\infty}^{\infty} |u|^k P_n(u)du < \int_{-\infty}^{\infty} |w|^k Q_m(w)dw,$$

or if

$$1/P^2 < 1/Q^2, \text{ or if } \sum_{i=1}^{n} p_i^2 < \sum_{j=1}^{m} q_j^2.$$

This shows that under this error law the probable value of any given positive power of the absolute error of the arithmetic mean is less than that of any other weighted mean. In this case $m=n$ and the weights are each equal to $1/n$.

If $p_i=1/m$ and $q_j=1/n$, where $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, n$, then the probable value of any given positive power of the absolute error of the first weighted mean, (here the real mean), is less than that of the second mean if $m<n$. This shows that the mean with a greater number of variates is the better. This is as it should be, or what would be expected. But the p's and q's can be such that even if $m>n$ the probable value of any given positive power of the absolute error of the first weighted mean is greater than that of the second weighted mean. The weighted mean with the most variates is not as good as that of the second with a smaller number of variates.

FOURIER SERIES IN THREE DIMENSIONS

By W. O. PENNELL, Bell Telephone Company, St. Louis

Introduction. The classical Fourier Series represents a function in a given interval and then repeats the same values in the next and subsequent intervals. In other words if $S(x)$ is the classical Fourier Series representing $f(x)$ in the interval $0<x<a$ then $S(x)=f(x-na)$ where n takes on the values $n=0, \pm1, \pm2, \cdots$ corresponding to the various intervals $na<x<(n+1)a$.

The author has[1] shown how a generalized Fourier Series $S_1(x)$ may be obtained representing a function in the intervals $na<x<(n+1)a$ as follows:

$$S_1(x) = b^n f(x - na)$$

where n takes on the values $n=0, \pm1, \pm2, \cdots$, and b is any real constant.

In this paper will be described a still more general Fourier Series $S_{11}(x)$ representing a function in the intervals $na<x<(n+1)a$ as follows:

$$S_{11}(x) = [b^n f(x - na)]_{n\phi}$$

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[1] A Generalized Fourier Series Representation of a Function, by W. O. Pennell—American Mathematical Monthly, Nov. 1930.

where corresponding to the above intervals n takes on the values $n=0, \pm 1, \pm 2, \cdots$, b is any real constant, and the subponent notation $n\psi$ denotes the rotation of the plane of the curve about the X axis through an angle $n\psi$ with the XY plane.

In other words $S_{11}(x)$ represents a curve in planes passing through the X axis as follows:

Interval	Value of $S(x)$	Angle with XY Plane of Plane of Curve
$0<x<a$	$f(x)$	0
$a<x<2a$	$bf(x-a)$	ψ
$2a<x<3a$	$b^2f(x-2a)$	2ψ
. . .	. . .	. . .
$-a<x<0$	$b^{-1}f(x+a)$	$-\psi$
$-2a<x<-a$	$b^{-2}f(x+2a)$	-2ψ
$-3a<x<-2a$	$b^{-3}f(x+3a)$	-3ψ
. . .	. . .	. . .

Deduction of Formula. Let $f(x)$ be a function which satisfies the condition necessary for representation by a classical Fourier Series in the interval $0<x<a$. Then in general $f(x)b^{-x/a}e^{-i\psi x/a}$, where b, a, and ψ are real constants, can be represented by a Fourier Series. Let such a series by denoted by $S(x)$. Then

$$S(x) = f(x - na)b^{-(x-na)/a}e^{-i\psi(x-na)/a} \tag{1}$$

when n takes on the values $n=0, \pm 1, \pm 2, \cdots$, corresponding to the intervals $na<x<(n+1)a$. From (1)

$$S(x)b^{x/a}e^{i\psi x/a} = f(x - na)b^n e^{i\psi n}. \tag{2}$$

Now plot $y=f(x-na)b^n e^{i\psi n}$ as follows:

Since x is limited to real values, it is plotted as usual along the X axis; but y is a vector or complex variable. The Y plane will be taken perpendicular to the X axis and passing through the Y axis. Values of y are plotted in this plane in the usual way, real values along the Y axis and imaginary values at right angles to it. Having chosen and located a value of x, the point corresponding to y is obtained by drawing from the point x, a vector parallel and equal to the y vector as located in the Y plane.

With these conventions in mind, it is readily seen that $e^{in\psi}$ is a rotor turning the plane of the curve about the X axis through an angle $n\psi$.

The rule for obtaining the space Fourier Series representing a function $b^n f(x-na)$ in planes passing through the X axis and making angles $n\psi$ with the XY plane where n has the values $0, \pm 1, \pm 2, \cdots$, corresponding to the intervals $na<x<(n+1)a$ can then be briefly stated as follows:

"Find the classical Fourier Series $S(x)$ for $f(x)b^{-x/a}e^{-i\psi x/a}$. Multiply the series so found by $b^{x/a}e^{i\psi x/a}$. The result $S(x)b^{x/a}e^{i\psi x/a}$ will be the space Fourier Series desired."

Some Characteristics of the Series. Since the space series is a Fourier Series multiplied by $b^{x/a}e^{i\psi x/a}$ it follows that it has the same kind of convergence as the Fourier Series and is subject to the same rules regarding term by term differentiation and integration as the classical series.

In the classical Fourier Series for $f(x)$, the value of $f(x)$ at a point of finite discontinuity, for example, $x=a$ is given by $\{f(a+0)+f(a-0)\}/2$. From its derivation, it is obvious that the space series follows the same rule except in this case $f(a+0)$ and $f(a-0)$ are vectors and the sum must be interpreted as the vector sum.

If the angle ψ is commensurable with 2π the curve, after a certain number of intervals periodically returns to a plane in which it was formerly located. If ψ and 2π are incommensurable the curve will never occupy the same plane a second time.

An Illustration. Find a Fourier Series representing a function in planes passing through the X axis as follows:

Interval	Value of Function	Angle of Plane of Curve With XY Plane
$0<x<a$	1	0
$a<x<2a$	b	ψ
$2a<x<3a$	b^2	2ψ
. . . . .		
$-a<x<0$	b^{-1}	$-\psi$
$-2a<x<-a$	b^{-2}	-2ψ
$-3a<x<-2a$	b^{-3}	-3ψ
. . . . .		

Going through the process described for the deduction of the series the result is:

$$(3)\quad y=\frac{b^{x/a-1}e^{i\psi(x/a-1)}(be^{i\psi}-1)}{\log_e b+i\psi}+\frac{2(be^{i\psi}-1)}{be^{i\psi}}\sum_{n=1}^{\infty}\frac{b^{x/a}e^{i\psi x/a}[\log_e b+i\psi]\cos\dfrac{2n\pi x}{a}+2n\pi b^{x/a}e^{i\psi x/a}\sin\dfrac{2n\pi x}{a}}{4n^2\pi^2+(\log_e b+i\psi)^2}.$$

If $b=1$ and $\psi=0$, (3) reduces to unity, which is the value of the function under these conditions.

At the point of discontinuity $x=a$, the value of $f(a+0)=be^{i\psi}$ and $f(a-0)=1$. So $\{f(a+0)+f(a-0)\}/2$ is $(be^{i\psi}+1)/2$. By substituting in (3) $x=a$ this is the value obtained.

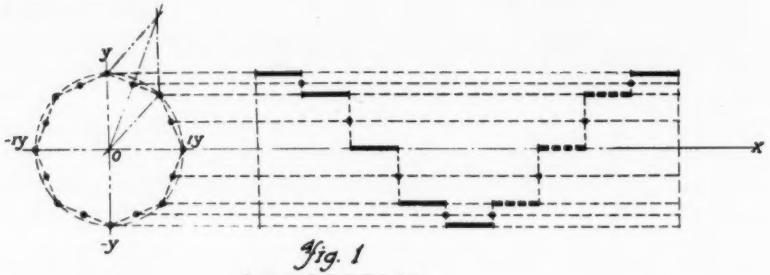

Fig. 1
PROJECTION

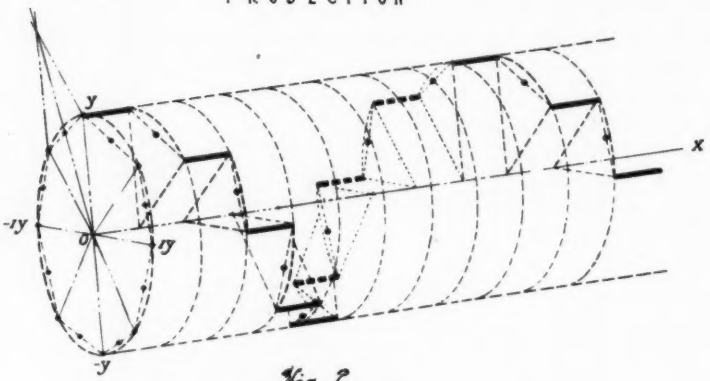

Fig. 2
PERSPECTIVE

In Fig. 1 is shown a projection upon the XY and upon the Y planes of the curve represented by (3) for $b=1$ and $\psi=\pi/4$. In Fig. 2 is shown an isometric view of the same.

The values at the points of discontinuity are shown by dots and it will be seen that they equal one half the vector sum of the values just before and just after the discontinuity.

Other examples can be obtained from the Author's paper[1] on "A Generalized Fourier Series" by substituting in the examples there solved, for b the vector $be^{i\psi}$.

Remarks on the Approximation Curves. It is interesting to note that in general the approximation curves, even for a single interval, are not confined to one plane, but are curves in space twisting about the ultimate curve they are to represent and coming nearer and nearer to it as the number of the approximation curves increases.

[1] American Mathematical Monthly, November, 1930.

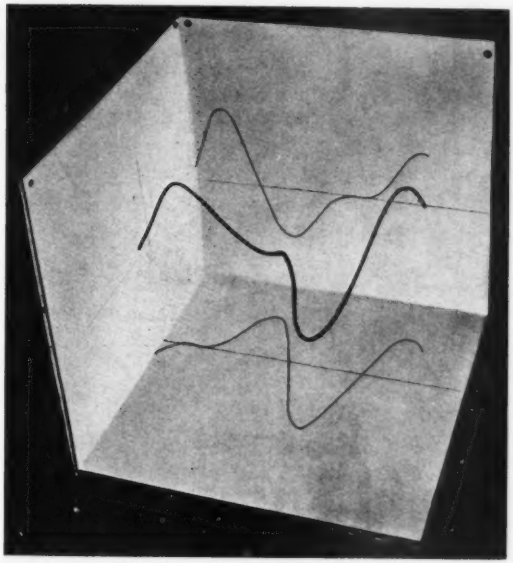

FIG. 3

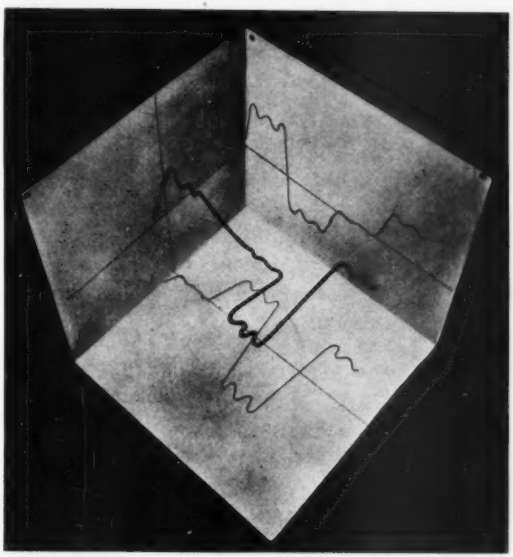

FIG. 4

Figures 3 and 4 respectively show the first and third approximation curve for the example illustrated in equation 3 when $b=1$ and the angle $\psi=5\pi/4$.

Various values of x and y were calculated from the formula and the corresponding projections on the XY and the $X\ i\ Y$ planes plotted. From these projections an actual model was constructed, the curve being represented by a wire bent into the proper shape. The model was then photographed together with the projections on the two planes mentioned above.

Acknowledgment. The author wishes to acknowledge his indebtness to Miss Elizabeth Harris who has calculated and plotted the approximation curves for a number of examples.

The author is also indebted to Mr. John Casey and Mr. W. F. Uphouse who constructed the models of the approximation curves and supervised the taking of the photographs.

VANISHING AGGREGATES OF DETERMINANTS AND THEIR RELATIONSHIPS

By R. A. BEAVER, New York State College for Teachers

In the years since Bézout's important work of 1779 a large number of papers have been published on the subject of vanishing aggregates of determinants. Many of the results given seem at first sight quite unrelated, but on closer observation prove to be intimately connected. The principal object of this paper is to point out which aggregates are fundamental and to show how the others are related to them.

§1

Deruyts (1882) and Muir (1888) proved the following theorem:[1]

THEOREM A: *If any two determinants A and B of the nth order be taken, and from these two sets of determinants be formed, namely, first, a set of ${}_nC_r$ determinants each of which is in r rows identical with A and in the remaining rows with B; and, second, a set of the same number of determinants each of which is in r columns identical with A and in the remaining columns with B, then the sum of the first set is equal to the sum of the second set.*

Denote this theorem by

$$R\begin{pmatrix} A & B \\ a & b \end{pmatrix} = C\begin{pmatrix} A & B \\ a & b \end{pmatrix}$$

where $a+b=n$.

In 1917 Metzler generalized it, giving the theorem[2]

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[1] Muir and Metzler, Theory of Determinants, (1929), §319. This work will hereafter be referred to as Text.

[2] Text, §320. Vanishing Aggregates, *Proceed. R. Soc. Edinburgh*, xxxvii, pp. 324–326.

THEOREM I:

$$R\begin{pmatrix} A & B \cdots K \\ a & b \cdots k \end{pmatrix} = C\begin{pmatrix} A & B \cdots K \\ a & b \cdots k \end{pmatrix}$$

where $a+b+\cdots+k=n$.

Theorems on linear relations between determinants have been given by Le Paige[1] (1880), Hammond (1881), Deruyts (1884), Dines (1925) and Muir (1930). Muir's theorem[2] is as follows:

If from an n-by-$(n+2)$ *array there be formed four determinants, namely,*

P	*by omitting the*	n*th*	*and*	$(n+2)$*th columns,*
Q	"	"	$(n+1)$*th and* 1*st*	" ,
Y	"	"	$(n-1)$*th and* $(n+2)$*th*	" ,
Z	"	"	n*th and*	$(n+1)$*th* " ,

then the sum of the determinants which each have $n-1$ *rows in common with* P *and one row in common with* Q *is equal to the sum of* Y *and* Z.

This theorem is seen to be a special case of *Theorem A* where $A=P$, $B=Q$ and $r=n-1$.

An example of Dines's theorem[3] when $n=3$ and $\alpha_1, \alpha_2, \alpha_3 = 0, 1, 2$ is

$$\begin{vmatrix} a_1 & a_2 & c_3 \\ b_1 & c_2 & 0 \\ c_1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_1 & c_2 & b_3 \\ b_1 & 0 & c_3 \\ c_1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} b_1 & a_2 & c_3 \\ c_1 & b_2 & 0 \\ 0 & c_2 & 0 \end{vmatrix} + \begin{vmatrix} b_1 & c_2 & a_3 \\ c_1 & 0 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}$$
$$+ \begin{vmatrix} c_1 & a_2 & b_3 \\ 0 & b_2 & c_3 \\ 0 & c_2 & 0 \end{vmatrix} + \begin{vmatrix} c_1 & b_2 & a_3 \\ 0 & c_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = 0.$$

This relation may be obtained by applying *Theorem I* to the determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \quad \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} c_1 & c_2 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Dines's general theorem and the other theorems mentioned above come out of *Theorem I* in similar fashion.

Theorems on vanishing aggregates of binary-products have been given by Bézout (1779), Monge (1809), Cauchy (1812), Desnanot (1819), Schweins

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[1] See Muir, T, History of Determiments, when explicit references are not given.

[2] Note on Sums of n-line Minors pertaining to an n-by-$(n+2)$ Array, *Trans. R. Soc. So. Africa*, xviii, pp. 301–303.

[3] On Certain Symmetric Sums of Determinants, *Am. Journ. of Math.*, xlvii, pp. 249–256.

(1825), Reiss (1829), Sylvester (1851), Hesse (1872), Muir (1879), (1913), and many others. These aggregates may be readily obtained by expanding certain zero determinants by Laplace's theorem, or as in the case of Schweins's, by expanding a certain determinant in two ways and equating the resulting aggregates, or again, as in the case of Desnanot's, by "extending" relations of these two types.

They may also be obtained by the use of *Theorem I*. Thus for Sylvester's theorem[1] of 1851: *The product of two determinants of the same order is equal to the sum of like products obtained from the original by interchanging k chosen columns of the one determinant with every set of k columns of the other in succession; the interchange of k columns with k columns being effected by interchanging the first column of the one set with the first column of the other, the second of the one with the second of the other, and so on.*

Let the two determinants of the theorem be $|a_{1n}|$ and $|b_{1n}|$, and let the k columns fixed upon be the first k columns of $|a_{1n}|$. Then the theorem is seen to be a special case of *Theorem A* where $r=n$, and A and B are respectively the determinants of order $2n$

$$\begin{vmatrix} 0\cdots 0 & a_{1,k+1}\cdots a_{1n} & b_{11}\cdots b_{1n} \\ 0\cdots 0 & a_{2,k+1}\cdots a_{2n} & b_{21}\cdots b_{2n} \\ \cdots & \cdots & \cdots \\ 0\cdots 0 & a_{n,k+1}\cdots a_{nn} & b_{n1}\cdots b_{nn} \\ 0\cdots 0\ 0\cdots 0 & 0\cdots 0 & \\ \cdots & \cdots & \\ 0\cdots 0\ 0\cdots 0 & 0\cdots 0 & \end{vmatrix}_{2n}$$

and

$$\begin{vmatrix} 0\cdots 0 & 0\cdots 0\ 0\cdots 0 & \\ \cdots & \cdots & \\ 0\cdots 0 & 0\cdots 0\ 0\cdots 0 & \\ a_{11}\cdots a_{1k} & 0\cdots 0 & b_{11}\cdots b_{1n} \\ a_{21}\cdots a_{2k} & 0\cdots 0 & b_{21}\cdots b_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1}\cdots a_{nk} & 0\cdots 0 & b_{n1}\cdots b_{nn} \end{vmatrix}_{2n},$$

the sign factor $(-1)^k$ cancelling out. The proof is similar when the k chosen columns are not the first.

Monge's example (1809) of a vanishing aggregate of ternary-products may also be obtained either from the Laplacian expansion of a zero determinant or

[1] Text, §137.

by applying *Theorem I* to three determinants. If $(rs\cdots)$ denote the array composed of the rth, sth, $\cdots$ columns of the array $\|a_{36}\|$ and $[t]$ denote an array of 3 rows and t columns of zeros, the three determinants may be represented by

$$A \equiv \begin{vmatrix} (1234) & [5] \\ [4] & [5] \\ [4] & [5] \end{vmatrix}, \quad B \equiv \begin{vmatrix} [4] & [4] & [1] \\ [4] & (1234) & [1] \\ [4] & [4] & [1] \end{vmatrix},$$

$$C \equiv \begin{vmatrix} [2] & [4] & [3] \\ [2] & [4] & [3] \\ (12) & [4] & (345) \end{vmatrix}.$$

That is, A is a determinant of order nine with a rectangle of three rows and four columns of a's in the upper left-hand corner, the remaining elements being all zeros.

§2

Two fundamental theorems[1] on aggregates of minors of general determinants of even order $2n$ were given by Metzler in 1901. These theorems cannot be briefly formulated; but the first, which we shall denote by *Theorem II*, expresses a sum of minors of order n as a sum of products of minors of order $h+g$ and $n-(h+g)$ respectively. When $h=k$ and $g=0$, *Theorem II* gives another[2] which we shall denote by *Theorem B*, an example of which is, in the umbral notation,

$$\begin{vmatrix} 1&2&3&4 \\ 5&6&7&8 \end{vmatrix} - \begin{vmatrix} 1&2&3&5 \\ 4&6&7&8 \end{vmatrix} + \cdots + \begin{vmatrix} 1&2&7&8 \\ 3&4&5&6 \end{vmatrix}$$

$$= \begin{vmatrix} 1&2 \\ 3&4 \end{vmatrix} \left\{ \begin{vmatrix} 5&6 \\ 7&8 \end{vmatrix} - \begin{vmatrix} 5&7 \\ 6&8 \end{vmatrix} + \begin{vmatrix} 5&8 \\ 6&7 \end{vmatrix} + \begin{vmatrix} 6&7 \\ 5&8 \end{vmatrix} - \begin{vmatrix} 6&8 \\ 5&7 \end{vmatrix} + \begin{vmatrix} 7&8 \\ 5&6 \end{vmatrix} \right\}$$

$$- \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \begin{vmatrix} 1&2 \\ 7&8 \end{vmatrix} \left\{ \begin{vmatrix} 3&4 \\ 5&6 \end{vmatrix} - \begin{vmatrix} 3&5 \\ 4&6 \end{vmatrix} + \begin{vmatrix} 3&6 \\ 4&5 \end{vmatrix} + \begin{vmatrix} 4&5 \\ 3&6 \end{vmatrix} - \begin{vmatrix} 4&6 \\ 3&5 \end{vmatrix} + \begin{vmatrix} 5&6 \\ 3&4 \end{vmatrix} \right\}.$$

The left member of this relation may be denoted by

$$\Sigma \begin{vmatrix} 12\overline{34} \\ \underline{5678} \end{vmatrix},$$

a notation due to Muir.

When $h=1$ and $k=n-1$ *Theorem B* reduces to a theorem given by Muir in 1900.

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[1] Text, §§316, 318. On Certain Aggregates of Determinant Minors, *Trans. Am. Math. Soc.*, ii, pp. 395–403.

[2] Text, §312.

If the original determinant from which the minors involved in the theorem are formed has the coaxial minor

$$\begin{vmatrix} k+1 & k+2 \cdots 2n \\ k+1 & k+2 \cdots 2n \end{vmatrix}$$

axisymmetric, the right side of the relation in *Theorem B* will vanish. Thus we have

$$\Sigma \begin{vmatrix} 1 & 2 & \cdots k & \overline{k+1 \cdots n} \\ \underline{n+1} & \underline{n+2} & \cdot & \cdot \quad \cdot \; \underline{2n} \end{vmatrix} = 0$$

for $k=1, 2, \cdots, n-1$.

When $k=n-1$ the above relation becomes Kronecker's of 1882.

In 1902 Muir showed that for an axisymmetric determinant there are additional relations among the minors of the type

$$\Sigma \begin{vmatrix} 12\overline{45} \\ \underline{3678} \end{vmatrix} = 0$$

which may be regarded as an extensional of the relation[1]

$$\Sigma \begin{vmatrix} 2\overline{45} \\ \underline{678} \end{vmatrix} = 0.$$

We see that

$$\Sigma \begin{vmatrix} 12\overline{34} \\ \underline{5678} \end{vmatrix} = \Sigma \begin{vmatrix} 123\overline{4} \\ \underline{5678} \end{vmatrix} + \left\{ \Sigma \begin{vmatrix} 124\overline{3} \\ \underline{5678} \end{vmatrix} + \begin{vmatrix} 1243 \\ 5678 \end{vmatrix} \right\} + \Sigma \begin{vmatrix} 12\overline{56} \\ 34\underline{78} \end{vmatrix}.$$

Since for axisymmetry the first, second and third sums above are each zero, we may write

$$\begin{vmatrix} 1234 \\ 5678 \end{vmatrix} = \Sigma \begin{vmatrix} 12\overline{56} \\ 34\underline{78} \end{vmatrix}.$$

Miss Barton[2] in 1926 proved the general theorem for which the above relation is an example, but Muir[3] had already given the theorem in 1902.

Runge in 1882 observed that in Kronecker's relation one of the invariable indices may be made the same as one of the variable. For example, in the Kronecker relation

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[1] Text, §529.

[2] Generalization of Kronecker's Relation among the Minors of a Symmetric Determinant, *Proc. Nat. Acad. Sci.*, xii, pp. 393–396.

[3] Aggregates of Minors of an Axisymmetric Determinant, *Philos. Magazine*, iii, pp. 410–416.

$$\Sigma \left| \begin{matrix} 123\overline{4} \\ \underline{5678} \end{matrix} \right| = 0$$

put $2=7$. The relation then reduces to an extensional of the Kronecker relation

$$\Sigma \left| \begin{matrix} 13\overline{4} \\ \underline{568} \end{matrix} \right| = 0.$$

Nanson (1903) has given relations which Metzler has shown to be obtainable from Kronecker relations by transposition of some of the terms. These relations are of the type

$$\Sigma \left| \begin{matrix} \overline{12}45 \\ \underline{3}678 \end{matrix} \right| = -\Sigma \left| \begin{matrix} 123\overline{4} \\ \underline{5}678 \end{matrix} \right| = \Sigma \left| \begin{matrix} 123\overline{6} \\ 457\underline{8} \end{matrix} \right|.$$

It is to be observed that the $2m$ numbers are divided into three groups, the first containing x, the second y and the third z of the $2m$ numbers, where $(x+y+z)=2m$ and in the first sigma the x numbers, in the second the y numbers, and in the third the z numbers vary while the others remain invariant in each case.[1]

Due to the fact that the whole determinant need not be axisymmetric for Kronecker's theorem, Muir (1888), (1911) has called attention to the desirability of looking at it from another standpoint. If we take with the array

$$\begin{matrix} a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \end{matrix}$$

the rows, one after another, of the zero-axial, axisymmetric determinant

$$\begin{vmatrix} 0 & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{45} & 0 & a_{56} & a_{57} & a_{58} \\ a_{46} & a_{56} & 0 & a_{67} & a_{68} \\ a_{47} & a_{57} & a_{67} & 0 & a_{78} \\ a_{48} & a_{58} & a_{68} & a_{78} & 0 \end{vmatrix}$$

and strike out the column containing the zero in each case, then the sum of the resulting determinants with signs alternately positive and negative is zero, for the aggregate is the Kronecker expression

$$\Sigma \left| \begin{matrix} 123\overline{4} \\ \underline{5678} \end{matrix} \right|$$

[1] Nanson, Minors of Axisymmetric Determinants, *Am. Journ. of Math.*, xxvii, pp. 69–76. Metzler, Variant Forms of Vanishing Aggregates of Minors of Axisymmetric Determinants, *Proceed. Roy. Soc. Edinburgh*, xxv, pp. 717–721. Text, §530.

Starting with this type of relation, Metzler builds up several vanishing aggregates of determinants.[1] It should be noticed that his relation[2] (4) in the text, (10) in the paper, is identical with that obtained by equating the two Kronecker expressions

$$\Sigma \left| \begin{matrix} 123\overline{4} \\ \underline{5678} \end{matrix} \right| = 0 = \Sigma \left| \begin{matrix} \overline{1234} \\ \underline{5678} \end{matrix} \right|$$

with respect to the axisymmetric determinant

$$\begin{vmatrix} 0 & \gamma & \beta & a_{21} & x & a_{22} & a_{23} & a_{24} \\ \gamma & 0 & \alpha & a_{31} & y & a_{32} & a_{33} & a_{34} \\ \beta & \alpha & 0 & a_{41} & z & a_{42} & a_{43} & a_{44} \\ a_{21} & a_{31} & a_{41} & 0 & \cdot & s & t & u \\ x & y & z & \cdot & 0 & a_{12} & a_{13} & a_{14} \\ a_{22} & a_{32} & a_{42} & s & a_{12} & 0 & r & q \\ a_{23} & a_{33} & a_{43} & t & a_{13} & r & 0 & p \\ a_{24} & a_{34} & a_{44} & u & a_{14} & q & p & 0 \end{vmatrix}$$

In 1902 Muir[3] gave a number of theorems on vanishing aggregates which depend on Kronecker's relation or extensionals of it.

Starting with any of the above mentioned types of relations between minors of axisymmetric determinants, relations between minors of persymmetric which are not true for axisymmetric determinants may be obtained.[4]

In 1911 Muir[5] gave two vanishing aggregates:

$$(1) \quad \begin{vmatrix} \theta_2 & \theta_3 & \theta_1 & \theta_4 \\ a & b & c & d \\ B & c & D & e \\ \alpha & \beta & \gamma & \delta \end{vmatrix} + \begin{vmatrix} a & B & \omega & \theta_2 \\ b & c & d & b' \\ c & D & e & c' \\ \beta & \gamma & \delta & b'' \end{vmatrix} - \begin{vmatrix} B & D & e & \theta_3 \\ a & c & d & a' \\ B & D & e & b' \\ \alpha & \gamma & \delta & a'' \end{vmatrix} + \begin{vmatrix} a & b & d & \theta_1 \\ a & b & d & c' \\ B & c & e & d' \\ \alpha & \beta & \delta & c'' \end{vmatrix}$$

$$- \begin{vmatrix} \omega & d & e & \theta_4 \\ a & b & c & b' \\ B & c & D & c' \\ \alpha & \beta & \gamma & b'' \end{vmatrix} = 0$$

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[1] Text, §§533–537. On Certain Determinant Relations, *Annals of Math.*, Second Series, xxvii, pp. 407–420.

[2] Text, §536.

[3] Vanishing Aggregates of Secondary Minors of a Persymmetric Determinant, *Trans. R. Soc. Edinburgh*, xl, pp. 511–533.

[4] Text, §§538–9.

[5] Cayley's Linear Relation between Minors of a Special Three-row Array, *Mess. of Math.*, xli, pp. 23–28.

and

$$(2)\quad \begin{vmatrix} \theta_2 & \xi & \theta_1 & \theta_4 \\ a & b & e & m \\ B & e & h & n \\ c & f & i & o \end{vmatrix} + \begin{vmatrix} a & B & \omega_1 & \theta_2 \\ b & e & m & s \\ e & h & n & v \\ f & i & o & w \end{vmatrix} + \begin{vmatrix} a & b & m & \theta_1 \\ a & b & m & v \\ B & e & n & y \\ c & f & o & z \end{vmatrix} - \begin{vmatrix} \omega_1 & m & n & \theta_4 \\ a & b & e & s \\ B & e & h & v \\ c & f & i & w \end{vmatrix}$$
$$+ \begin{vmatrix} \theta_3 & \theta_6 & \xi & \theta_5 \\ a & e & h & m \\ B & h & k & n \\ c & i & l & o \end{vmatrix} + \begin{vmatrix} a & B & \omega_2 & \theta_3 \\ e & h & m & q \\ h & k & n & s \\ i & l & o & t \end{vmatrix} - \begin{vmatrix} B & k & n & \theta_6 \\ a & h & m & p \\ B & k & n & q \\ c & l & o & r \end{vmatrix} - \begin{vmatrix} \omega_2 & m & n & \theta_5 \\ a & e & h & q \\ B & h & k & s \\ c & i & l & t \end{vmatrix} = 0.$$

Muir's relations as he gave them had b in place of B but the more general relations which we have written are still true.

That the relation

$$(3)\quad \begin{vmatrix} \theta_2 & \theta_3 & \theta_1 & \theta_4 \\ a & b & c & d \\ B & c & D & e \\ \alpha & \beta & \gamma & \delta \end{vmatrix} - \begin{vmatrix} \theta_2 & a & B & \omega \\ b' & b & c & d \\ c' & c & D & e \\ b'' & \beta & \gamma & \delta \end{vmatrix} + \begin{vmatrix} \theta_3 & a & c & d \\ b' & a & c & d \\ c' & B & D & e \\ b'' & \alpha & \gamma & \delta \end{vmatrix} - \begin{vmatrix} \theta_1 & B & c & e \\ b' & a & b & d \\ c' & B & c & e \\ b'' & \alpha & \beta & \delta \end{vmatrix}$$
$$+ \begin{vmatrix} \theta_4 & \omega & d & e \\ b' & a & b & c \\ c' & B & c & D \\ b'' & \alpha & \beta & \gamma \end{vmatrix} = 0$$

is true as being an example of Kronecker's theorem may be seen on taking the sum of the determinants resulting from taking with the array

$$\begin{matrix} b' & a & b & c & d \\ c' & B & c & D & e \\ b'' & \alpha & \beta & \gamma & \delta \end{matrix}$$

the rows, one after another, of the determinant

$$\begin{vmatrix} 0 & \theta_2 & \theta_3 & \theta_1 & \theta_4 \\ \theta_2 & 0 & a & B & \omega \\ \theta_3 & a & 0 & c & d \\ \theta_1 & B & c & 0 & e \\ \theta_4 & \omega & d & e & 0 \end{vmatrix}$$

and striking out the column containing the zero in each case.

The third determinant in (3) is seen to be equal to

$$\begin{vmatrix} \theta_3 - b' & 0 & 0 & 0 \\ b' & a & c & d \\ c' & B & D & e \\ b'' & \alpha & \gamma & \delta \end{vmatrix} = \begin{vmatrix} \theta_3 - b' & 0 & 0 & 0 \\ a' & a & c & d \\ b' & B & D & e \\ a'' & \alpha & \gamma & \delta \end{vmatrix} = \begin{vmatrix} \theta_3 & B & D & e \\ a' & a & c & d \\ b' & B & D & e \\ a'' & \alpha & \gamma & \delta \end{vmatrix}.$$

Similarly, the fourth determinant is seen to be equal to

$$\begin{vmatrix} \theta_1 & a & b & d \\ c' & a & b & d \\ d' & B & c & e \\ c'' & \alpha & \beta & \delta \end{vmatrix}$$

Making these substitutions in (3) we obtain Muir's relation (1).

The following two relations are true as being examples of (1):

$$(4) \quad \begin{vmatrix} \theta_2 & \xi & \theta_1 & \theta_4 \\ a & b & e & m \\ B & e & h & n \\ c & f & i & o \end{vmatrix} + \begin{vmatrix} a & B & \omega_1 & \theta_2 \\ b & e & m & s \\ e & h & n & v \\ f & i & o & w \end{vmatrix} - \begin{vmatrix} B & h & n & \xi \\ a & e & m & a' \\ B & h & n & s \\ c & i & o & a'' \end{vmatrix} + \begin{vmatrix} a & b & m & \theta_1 \\ a & b & m & v \\ B & e & n & y \\ c & f & o & x \end{vmatrix}$$

$$- \begin{vmatrix} \omega_2 & m & n & \theta_4 \\ a & b & e & s \\ B & e & h & v \\ c & f & i & w \end{vmatrix} = 0,$$

$$(5) \quad \begin{vmatrix} \theta_3 & \theta_6 & \xi & \theta_5 \\ a & e & h & m \\ B & h & k & n \\ c & i & l & o \end{vmatrix} + \begin{vmatrix} a & B & \omega_2 & \theta_3 \\ e & h & m & q \\ h & k & n & s \\ i & l & o & t \end{vmatrix} - \begin{vmatrix} B & k & n & \theta_6 \\ a & h & m & p \\ B & k & n & q \\ c & l & o & r \end{vmatrix} + \begin{vmatrix} a & e & m & \xi \\ a & e & m & s \\ B & h & n & d' \\ c & i & o & c'' \end{vmatrix}$$

$$- \begin{vmatrix} \omega_2 & m & n & \theta_5 \\ a & e & h & q \\ B & h & k & s \\ c & i & l & t \end{vmatrix} = 0.$$

We observe, on expanding by minors of elements in the fourth column, that the third determinant in (4) is equal to the fourth determinant in (5). If we add relations (4) and (5) these equal determinants cancel and we have Muir's relation (2).

§3

Metzler's second fundamental theorem of 1901 we shall denote by *Theorem III*. It expresses a sum of minors of order n as a sum of products of minors of order $n-k$ and k respectively.

When $k=1$ the theorem reduces to one given by Muir in 1900.

If the original determinant is centrosymmetric, the right-hand side of the relation given in *Theorem III* vanishes, giving a general linear relation[1] connecting minors of order n of a centrosymmetric determinant of order $2n$. The type $k=1$ was first pointed out by Muir in 1888.

§4

It appears, then, that the theorems on aggregates come out by specialization from *Theorems I, II, III*. This seems to be true of all known aggregates except one type connected with circulants which has been considered by Muir and Metzler.[2] For example, since the determinant

$$\begin{vmatrix} 1 & a+b+c & d \\ 1 & b+c+d & a \\ 1 & c+d+a & b \end{vmatrix}$$

is identically zero, we have

$$\begin{vmatrix} 1 & a & d \\ 1 & b & a \\ 1 & c & b \end{vmatrix} + \begin{vmatrix} 1 & b & d \\ 1 & c & a \\ 1 & d & b \end{vmatrix} + \begin{vmatrix} 1 & c & d \\ 1 & d & a \\ 1 & a & b \end{vmatrix} = 0.$$

This idea of basing the vanishing of an aggregate on an identity may be carried further. In any vanishing aggregate of minors of a determinant the coefficient of any letter must be a vanishing aggregate of lower order or else it must be identically zero. Thus in the Kronecker relation

$$\Sigma \begin{vmatrix} 1\bar{2} \\ \underline{34} \end{vmatrix} = 0$$

the coefficient of a_{13} is $a_{24}-a_{42}$ which may be considered the simplest type of Kronecker aggregate as its vanishing depends on axisymmetry. But the coefficient of a_{23} is $-a_{14}+a_{14}$ which is identically zero. Thus fundamentally the vanishing of any aggregate of minors depends on the construction of the determinant whose minors are involved and upon identities. Not only can we see the truth of such a relation by breaking it down in this manner, but we can build up vanishing aggregates by starting with ones of lower order or with an identity. Relation (3), §2, for example, may be built up by starting with an identity.

[1] Text, §524.

[2] Text, §540.

By the use of *Theorems I, II, III* we may deduce not only known theorems but also new ones worthy of note. Using the notation of the last paragraph of §1 with respect to the array

$$\begin{array}{ccccccccc} a_1 & b_1 & c_1 & r_1 & s_1 & t_1 & x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 & r_2 & s_2 & t_2 & x_2 & y_2 & z_2 \\ a_3 & b_3 & c_3 & r_3 & s_3 & t_3 & x_3 & y_3 & z_3 \end{array}$$

we find that upon applying *Theorem I* to the three determinants

$$\begin{vmatrix} (12) & [4] & (789) \\ [2] & [4] & [3] \\ [2] & [4] & [3] \end{vmatrix}, \quad \begin{vmatrix} [2] & [4] & [3] \\ [2] & (3456) & [3] \\ [2] & [4] & [3] \end{vmatrix}, \quad \begin{vmatrix} [3] & [6] \\ [3] & [6] \\ [3] & (456789) \end{vmatrix},$$

there results the relation

$$\begin{aligned} |a_1b_2c_3| \cdot |r_1s_2t_3| \cdot |x_1y_2z_3| &= |a_1b_2x_3| \cdot |r_1s_2c_3| \cdot |t_1y_2z_3| \\ + |a_1b_2x_3| \cdot |r_1c_2t_3| \cdot |s_1y_2z_3| &+ |a_1b_2x_3| \cdot |c_1s_2t_3| \cdot |r_1y_2z_3| \\ + |a_1b_2y_3| \cdot |r_1s_2c_3| \cdot |x_1t_2z_3| &+ |a_1b_2y_3| \cdot |r_1c_2t_3| \cdot |x_1s_2z_3| \\ + |a_1b_2y_3| \cdot |c_1s_2t_3| \cdot |x_1r_2z_3| &+ |a_1b_2z_3| \cdot |r_1s_2c_3| \cdot |x_1y_2t_3| \\ + |a_1b_2z_3| \cdot |r_1c_2t_3| \cdot |x_1y_2s_3| &+ |a_1b_2z_3| \cdot |c_1s_2t_3| \cdot |x_1y_2r_3|, \end{aligned}$$

which expresses the product of three determinants of order three in terms of like products.

To obtain a similar expression for the product of four determinants of order three *Theorem I* would be applied to four determinants of order twelve, nine of the rows of each of which would contain only zero elements.

The general theorem to which we are led is a generalization of Sylvester's theorem of 1851 referred to above and may be enunciated thus:

The product of p determinants of order n is equal to the sum of like products obtained from the original as follows: Replace a definite set of k columns of the first determinant by any set of k columns of the pth. Replace any set of k columns of the second by those columns displaced from the first, and so on, finally replacing the columns taken from the pth determinant by those displaced from the $(p-1)$th. The replacement of k columns by k columns is effected by replacing the first column of the one set with the first column of the other, the second of the one with the second of the other, and so on.

FLOW OF HEAT IN A DISC HEATED BY A GAS STREAM

By R. L. PEEK, JR., Bell Telephone Laboratories

A problem in the flow of heat, for which the solution does not appear to be recorded elsewhere, arises when a gas passes through a cylindrical vessel which is broken up into a number of chambers by flat floors or partitions perpendicular

to the axis of the vessel. This is the case in various processes in which a solid material passes from hearth to hearth in a vertical cylindrical vessel of this type, while a stream of gas flows in the opposite direction. The heat problem in question relates to the amount of heat which is taken up from the gas and flows outward through the circular partition or floor, assuming that the periphery of the latter is integral with the outside of the cylindrical vessel and is cooled so as to be at a constant temperature.

The problem is not difficult, and the solution is readily obtained by the methods discussed in courses dealing with the partial differential equations of mathematical physics. It is believed, however, that it may be of interest to teachers of such courses since it involves the expansion of a function in an orthogonal trigonometric series whose terms are not harmonic. Examples of this sort having genuine physical significance and yet simple enough to be used as problem material are rather rare.

It is assumed that the heat transfer from the gas to the plate is proportional to the temperature difference between them, i.e. that:

$$K\frac{\partial\theta}{\partial x} = \pm\, k(\theta_1 - \theta) \text{ when } (x = \pm\, a), \tag{1}$$

where a is the half thickness of the plate, K is the conductivity of the plate, θ its (variable) temperature, θ_1 the (constant) temperature of the gas, and k the coefficient of heat transfer. Either side of equation (1) represents the heat entering a unit area of either surface of the plate. The problem is then to solve the heat equation for a steady state:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{dr^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} = 0 \tag{2}$$

subject to the boundary conditions given by equation (1) and the condition that the edge of the plate be at the constant temperature θ_0, i.e.:

$$\theta = \theta_0 \text{ when } r = R. \tag{3}$$

A solution to equation (2) may be written in the form:

$$\theta = \theta_1 + \sum A_m J_0\left(\frac{im\pi}{a}\, r\right)\cos\left(\frac{m\pi}{a}\, x\right). \tag{4}$$

By substitution of this expression in equation (1) it is seen that this condition will be satisfied if the values of m appearing in the different terms of the series are given by $m\pi/a = \lambda_m$ where λ_m is a root of the equation:

$$\tan(a\lambda) = p/\lambda, \tag{5}$$

where $p = k/K$. As there are an infinite number of such roots, there may be an infinite number of terms in the series. The remaining condition, equation (3),

may be satisfied by choosing the value of A_m for each term so that the series is (for $r=R$) an expansion of $\theta_0-\theta_1$ from $x=0$ to $x=a$, i.e.:

$$\theta_0 - \theta_1 = \sum A_m J_0(i\lambda_m R)\cos(\lambda_m x). \tag{6}$$

An expansion of this sort, where the values of λ_m are roots of equation (5), may be readily accomplished, as the series is orthogonal.[1] When the function to be expanded is a constant, as in this case, the coefficients of the cosine terms are readily evaluated and yield:

$$A_m J_0(i\lambda_m R) = \frac{2(p^2+\lambda_m^2)}{a\lambda_m^2 + ap^2 + p}\,\frac{\sin a\lambda_m}{\lambda_m}(\theta_0-\theta_1). \tag{7}$$

The total quantity of heat flowing out through the edge of the disc per unit time (Q/t) is given by the quantity $-K(\partial\theta/\partial r)\partial x dr$ integrated over this surface, or

$$\begin{aligned} Q/t &= 4\pi RK(\theta_0-\theta_1)\int_0^a \sum A_m i J_1(i\lambda_m R)\cos(\lambda_m x)dx \\ &= -\,8\pi RK(\theta_1-\theta_0)\sum \frac{p^2}{\lambda_m(a\lambda_m^2+ap^2+p)}\,\frac{iJ_1(i\lambda_m R)}{J_0(i\lambda_m R)} \\ &= 8\pi RKS(\theta_1-\theta_0) \end{aligned} \tag{8}$$

where S is written for the absolute value of the summation appearing in the expression, which has always a negative value.

To evaluate the series S, it is first necessary to determine the value of λ_m for each term included in the computation: in general two or three terms suffice to evaluate S to four significant figures or better. Values of k and K are given in various tables. The former is an empirical constant dependent on the gas velocity and the state of the gas, for which values are given in various engineering texts and in the International Critical Tables. (This work, among others, gives values of K.) With k, K, and a known, values of λ_m are obtained by solution of equation (5).

For engineering purposes, equation (5) would probably be solved graphically, as by determining the intersections of $y=ap/x$ with $y=\tan x$; which would give values of $a\lambda_m$. For greater accuracy the roots could be obtained from the series expansion:

$$a\lambda_{n+1} = n\pi\left[1+\frac{ap}{(n\pi)^2}-\frac{(ap)^2}{(n\pi)^4}-\left\{\frac{1}{3(n\pi)^4}-\frac{2}{(n\pi)^6}\right\}(ap)^3+\cdots\right], \tag{9}$$

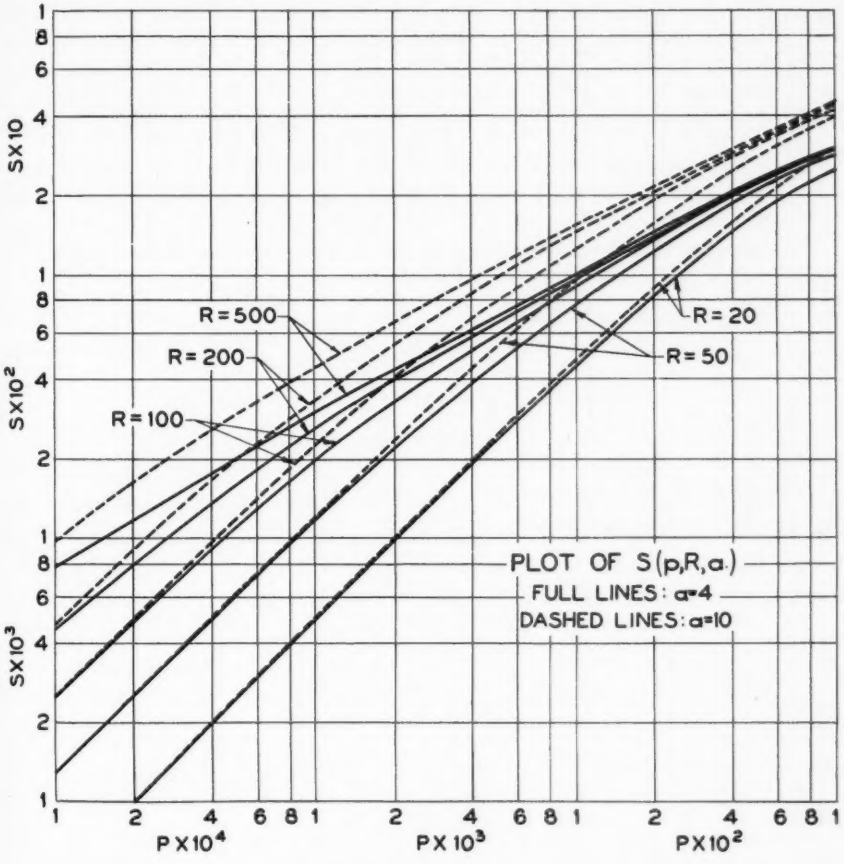
which applies to all roots except the first $(n=0)$.

Equation (9) is obtained by writing equation (5) as $ap=(n\pi+q_n)\tan q_n$, expanding the right-hand side in a power series, and subjecting the series to inversion. The same process can be used to obtain a series for the first root $(a\lambda_1)$, except that in this case the inversion of the series will lead to fractional powers of ap.[2]

[1] Carslaw, Mathematical Theory of Heat Conduction, 2nd Ed., 1921, p. 132.

[2] The writer is indebted to Dr. T. C. Fry of these Laboratories for the development of equation (9).

For engineering purposes a solution to a problem of this sort should, when possible, be reduced to a form less cumbrous than that of equation (8). In this case, curves were prepared showing the quantity S of equation (8) as a function of p, R and a. For selected pairs of values of R and a, curves of S vs. p were plotted on logarithmic paper. The curves were grouped in families for which a



had a constant value but in which R varied from curve to curve. Two such families are shown in Fig. 1, that for which $a = 4$ cms. being shown in full lines, while that for which $a = 10$ cms. is shown in dashed lines. It was found that with two additional groups ($a = 1.0$ cm. and $a = 2.0$ cms.) values of S for all values of a and R in the range indicated could be obtained by interpolation with sufficient accuracy.

In the actual problem each plate had a central orifice for the passage of gas

from one chamber to the other. As the temperature gradient near the center will be low the error introduced by ignoring the effect of such an orifice is small. In practice also the gas velocity and hence the value of k will not be uniform over the surface of the plate. Even if such an effect could be considered in developing the solution, the data required for its evaluation would not be likely to be available, and it is worthy of consideration only in estimating the accuracy of the result.

THE PRODUCT OF A CIRCULANT MATRIX AND A SPECIAL DIAGONAL MATRIX

By J. WILLIAMSON, Johns Hopkins University

Introduction. In the following paper we discuss an n-rowed square matrix

$$B = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ \omega a_{n-1} & \omega a_0 & \omega a_1 & \cdots & \omega a_{n-2} \\ \omega^2 a_{n-2} & \omega^2 a_{n-1} & \omega^2 a_0 & \cdots & \omega^2 a_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega^{n-1} a_1 & \omega^{n-1} a_2 & \omega^{n-1} a_3 & \cdots & \omega^{n-1} a_0 \end{pmatrix},$$

where ω is a primitive nth root of unity.[1] This matrix B is the product ΩA of the diagonal matrix

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}$$

and the circulant matrix[2]

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}.$$

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[1] That is a root of the equation $x^n - 1 = 0$ but is not a root of any equation $x^m - 1 = 0$, where $m < n$.

[2] The determinant of a circulant matrix is known. See T. Muir, *The Theory of Determinants*, page 189.

We prove that the nth power of the matrix B is the product of the determinant of A by the unit matrix. The proof is interesting and is peculiar, in as much as it is easier, when n the order of the matrix is a prime. For this reason the proof is given first, when n is a prime, and is later modified to suit the case when n is composite.

§1. *The order n a prime.* The elements in any one row or column of the matrix A are permutations of the n elements $a_0, a_1, \cdots, a_{n-1}$, and, if for brevity we write $a_i=i$, then it is apparent that the element in the ith row and jth column of A is $j-i$ or $n+j-i$ according as $i \leqq j$ or $i>j$, and may therefore be denoted by $j-i$, where $j-i=a_k$ if $j-i \equiv k \bmod n$, $0 \leqq k<n$. We now prove the following lemmas.

LEMMA 1. *The matrices of the m-rowed principal minors of A formed from the i_1th, i_2th, $\cdots$, i_mth rows and columns and formed from the j_1th, j_2th, $\cdots$, j_mth rows and columns are equal, if and only if*

$$i_r - i_s \equiv j_r - j_s \bmod n, \quad r, s = 1, 2, \cdots, m. \tag{1}$$

For the elements in the sth row and the rth column of the two matrices are respectively i_r-i_s and j_r-j_s. But (1) is true for all values of r and s, if

$$i_r - i_{r-1} \equiv j_r - j_{r-1} \bmod n, \quad r = 2, 3, \cdots, m. \tag{2}$$

Let $j+k_0, j+k_1, \cdots, j+k_{m-1}$ be m integers, $m<n$, where $k_1 \not\equiv k_r \bmod n$ if $i \neq r$; then the residues of these m integers modulo n are distinct and form a combination of the integers $0, 1, 2, \cdots, n-1$, which we shall denote by $M(k,j)$. By giving j the values $0, 1, 2, \cdots, n-1$ in turn we have n such combinations $M(k,0), M(k,1), \cdots, M(k,n-1)$, which form a set denoted by M_k.

LEMMA 2. *If m is relatively prime to n, the combinations of the set M_k are distinct.*

Let the members of the combination $M(k,r)$ coincide in some order with the members of the combination $M(k,s)$. Then certainly

$$mr + k_0 + k_1 + \cdots + k_{m-1} \equiv ms + k_0 + k_1 + \cdots + k_{m-1} \bmod n.$$

Hence

$$mr \equiv ms \bmod n,$$

or, since m and n are relatively prime,

$$r \equiv s \bmod n,$$

and, since both r and s are less then n,

$$r = s.$$

Accordingly this proves the lemma.

LEMMA 3. *If two sets M_k and M_t have one combination in common, then the combinations of M_k are the same as those of M_t except for a re-arrangement in order.*

Let $M(k,j)$ coincide with $M(t,r)$. Then for every value of $i=0, 1, 2, \cdots, m-1$, there must be an integer p $(0 \leqq p < m)$, such that $j+k_1 \equiv r+t_p \bmod n$. But the combinations of the set M_k are $M(k,j+u)$, $u=0, 1, \cdots, n-1$, where $M(k,j+u)$ is identical with $M(k,j+u-n)$ if $j+u \geqq n$. Hence every integer in the combination $M(k,j+u)$ is congruent to an integer of the form $j+k_i+u$ and is therefore congruent to $r+t_p+u$ modulo n. Accordingly the combination $M(k,j+u)$ coincides with the combination $M(t,r+u)$ and so the sets M_k and M_t coincide.

If now we consider two sets to be the same, if the combinations of the one are identical with the combinations of the other, and notice that every combination of m integers chosen from the integers $0, 1, \cdots, n-1$, must occur in at least one set, we have the following result.

LEMMA 4. *If m is an integer less than n and if n and m are relatively prime, the combinations of the integers $0, 1, \cdots, n-1$, taken m at a time, can be arranged into distinct sets, where each set contains n distinct combinations.*

Corollary. Since the number of combinations of n things m at a time is $\binom{n}{m}$, the number of distinct sets is $\binom{n}{m}/n$.

LEMMA 5. *Let $t_0, t_1, \cdots, t_{n-1}$, be the residues modulo n of the sums of the integers in the combinations $M(k, 0)$, $M(k, 1)$, $\cdots$, $M(k, n-1)$ respectively. Then $t_0, t_1, \cdots, t_{n-1}$ form a complete residue system modulo n, if m is relatively prime to n.*

For,

$$t_1 \equiv i + k_0 + i + k_1 + \cdots + i + k_{m-1} \bmod n,$$
$$\equiv mi + g, \text{ where } g = k_0 + k_1 + \cdots + k_{m-1}.$$

But if m is relatively prime to n, the integers $mi+g$, $i=0, 1, \cdots, n-1$, form a complete residue system modulo n.[1]

We are now in a position to prove the following theorem for the case in which n is a prime number.

THEOREM 1. *The nth power of the matrix B is equal to the product of the determinant of A by the unit matrix*; or

$$B^n = (-1)^{n-1} |B| E = |A| E,$$

where $|A|$ denotes the determinant of A and E is the unit matrix.

The characteristic equation of B is $|B-\lambda E|=0$, or $\lambda^n - b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + (-1)^n b_n = 0$, where b_m is the sum of the m-rowed principal minors of B. If we let the m-rowed principal minor formed from the $(j+k_i)$-th rows and

[1] L. E. Dickson, *Introduction to the Theory of Numbers*, Theorem 9, p. 8.

columns of B, $(i=0, 1, \cdots, m-1)$ correspond to the combination $M(k, j)$, by lemma 4 the $\binom{n}{m}$ m-rowed principal minors of B may be divided into $\binom{n}{m}/n$ sets, each set consisting of n minors. But by lemma 1 the minors corresponding to the set M_k differ from each other only by a power of ω. In fact, if Δ_k denote the value of the m-rowed principal minor of A corresponding to any combination of the set M_k, the value of the m-rowed principal minor of B corresponding to the combination $M(k, j)$ is, with the notation of lemma 5, $\omega^{ti}\Delta_k$. Therefore the sum of the n principal minors determined by the set of combinations M_k is

$$\begin{aligned}&\Delta_k(\omega^{t0} + \omega^{t1} + \cdots + \omega^{t(n-1)})\\ &= \Delta_k(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) \text{ by lemma 5,}\\ &= 0.\end{aligned}$$

Hence the coefficients b_m of the characteristic equation of B vanish for all values of $m=1, 2, \cdots, n-1$. Accordingly the characteristic equation of B is

$$\lambda^n + (-1)^n |B| = 0.$$

But every square matrix satisfies its own characteristic equation[1] and therefore

$$B^n = (-1)^{n-1}|B|E, \text{ where } E \text{ is the unit matrix.}$$

But, as we shall see later, $|B| = (-1)^{n-1}|A|$, and so the theorem is proved, when n is a prime.

§2. *The order n composite.* If n is not a prime number, for some value of m less than n, m and n will not be relatively prime and accordingly lemma 2 will no longer be true and the above proof must be modified.

If m and n are not relatively prime, let p be the greatest common divisor of m and n so that

$$m = pq,\ n = pt,\ q \text{ and } t \text{ are relatively prime.} \tag{3}$$

Since the truth of lemma 3 does not depend on the fact that m and n are relatively prime, we can still divide the $\binom{n}{m}$ combinations of n integers m at a time into sets, where each combination occurs in one and only one set, but, since lemma 2 is no longer true, the same combination may be repeated in its set. We can however prove the subsidiary lemma.

LEMMA 6. *In any set M_k the t combinations*

$$M(k, rt),\ M(k, rt+1),\ \cdots,\ M(k, rt+t-1) \textit{ are distinct.}$$

For, if the combination $M(k, rt+f)$ coincides with the combination $M(k, rt+s)$, then as in lemma 2, $mf \equiv ms \bmod n$ and by (3) $qf \equiv qs \bmod t$. But since q and t are relatively prime and f and s are both less than t, this last result is only true if $f=s$.

Thus the set M_k can be divided into p subsets consisting of t combinations each, the combinations in each subset being distinct. Further, if one combina-

[1] L. E. Dickson, *Modern Algebraic Theories*, page 48.

tion appear in two subsets, these subsets coincide. The proof is similar to that of lemma 3. Let the combination $M(k, rt+f)$ coincide with the combination $M(k, st+d)$. Then certainly

$$mrt = mf \equiv mst + md \bmod n,$$

$$mf \equiv md \bmod n \text{ by (3)},$$

$$qf \equiv qd \bmod t,$$

and therefore $f=d$. Hence the combination $M(k, rt+f)$ coincides with the combination $M(k, st+f)$ and as a result the combination $M(k, rt+i)$ coincides with the combination $M(k, st+i)$, for all values of i, $0 \leqq i < t$. Accordingly it is possible to divide the $\binom{n}{m}$ combinations of n integers m at a time into $\binom{n}{m}/t$ distinct sets of t combinations each, where each set consists of combinations of the type, $M(k, rt+i)$, $i=0, 1, \cdots, t-1$.

As before the sum of the principal minors of B corresponding to this subset is

$$\Delta_k \sum_{i=0}^{t-1} \omega^{k_0+k_1+\cdots+k_{m-1}+mrt+mi},$$

$$= \omega^g \Delta_k \sum_{i=0}^{t-1} \omega^{mi}, \; g = k_0 + k_1 + \cdots + k_{m-1} + mrt.$$

We now wish to show that $\sum_{i=0}^{t-1} \omega^{mi} = 0$, where ω is a primitive nth root of unity.

By (3) the set of integers $0, q, 2q, \cdots, (t-1)q$ forms a complete residue system modulo t and is therefore congruent modulo t to the set $0, 1, 2, \cdots, t-1$ in some order. Accordingly the set $0, m, 2m, \cdots, (t-1)m$ is congruent modulo n to the set $0, p, 2p, \cdots, (t-1)p$ in some order. Therefore,

$$\sum_{i=0}^{t-1} \omega^{mi} = \sum_{i=0}^{t-1} \omega^{pi} = \sum_{i=0}^{t-1} (\omega^p)^i = 0,$$

for, if ω is a primitive nth root of unity ω^p satisfies the equation $x^{t-1}+x^{t-2} + \cdots + x+1=0$. Thus the coefficients b_m of λ^m in the characteristic equation of B vanish for $m=1, 2, \cdots, n-1$ also when m is composite, and for all possible values of n

$$B^n = (-1)^{n-1} |B| E.$$

But, if n is odd,

$$|B| = \omega^{n(n-1)/2} |A| = (\omega^n)^{(n-1)/2} |A| = |A|.$$

On the other hand, if n is even,

$$|B| = \omega^{n(n-1)/2} |A| = (\omega^{n/2})^{n-1} |A| = (-1)^{n-1} |A| = -|A|.$$

Accordingly, for all values of n, $B^n = |A| E$ and theorem 1 is proved.

If

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

and $\omega_1, \omega_2, \cdots, \omega_n$ are the nth roots of unity[1]

$$|A| = \prod_{i=1}^{n} f(\omega_i) = k^n.$$

Then $(B/k)^n = E$, and the latent roots[2] of B/k are $\omega_1, \cdots \omega_n$ i.e. are 1, ω, $\cdots, \omega^{n-1}$. But $B = \Omega A$, and so there exists a non-singular matrix X such that

$$X\Omega A/kX^{-1} = \Omega, \text{ or } A/k = \Omega^{-1}X^{-1}\Omega X.$$

This last result is simply an alternative statement of theorem 1.

Moreover the matrices $\omega^i B$, $i = 0, 1, \cdots, n-1$, are permutable with each other and the elementary symmetric functions, of degrees 1, 2, $\cdots, n-1$ in these n matrices, are all zero and consequently are equal to the corresponding elementary symmetric functions of the latent roots of B. Accordingly we have proved;

THEOREM 2. *The matrices $\omega^i B$ form a complete set of conjugate matrices[3] all of which satisfy the characteristic equation of B.*

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

DEFINITIONS OF PROBABILITY[4]

By B. H. CAMP, Wesleyan University

A good definition of probability should have at least the following characteristics: it should be an idealization of one or more of the notions commonly associated with the term probability; it should admit proof of the sum and product theorems; and, taken together with these theorems, it must not lead to an inconsistency. Let us consider first the case in which there are but a finite number of possibilities or elements in the universe sampled. It is well understood that a satisfactory definition can be devised, provided one can define the words "equally probable." Some authors choose to escape this difficulty by asserting that this phrase is not definable, and that what is meant by it is understood intuitively. This attitude is logically defensible, but it is a defeatist attitude,

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[1] T. Muir, loc. cit.

[2] Bôcher, *Introduction to Higher Algebra*, page 283.

[3] Taber, American Journal of Mathematics, Vol. 13, p. 159.

[4] Part of a paper presented at a meeting of the American Statistical Association, December, 1931, under the title, "*Some fundamental concepts in statistics.*" This part has special reference to some remarks on probability in this *Monthly* by W. A. Wilson, vol. 38 (1931), pp. 578, 579.

for one cannot avoid the suspicion that the notion is declared to be intuitive because the definition is difficult.

There are three general problems to which the term probability is sometimes applied, and it is useful to distinguish between them before attempting a definition. So-called a priori probability has to do with the problem of piercing the veil which separates the future from the present, a posteriori probability with the problem of piercing the veil which separates the unknown past from the present, and there is finally the problem of passing from the known at any instant to the unknown at the same instant, as in the query: Are there mammals on Mars? With these last two problems we shall not be concerned here. They do not always have a meaning, and when they do, it is expressible in terms of a priori probability. The fundamental question with which we shall have to do resolves itself, therefore, into this: *What is meant by the statement that heads and tails are equally probable when a coin is tossed?*

There appear to be but two popular notions of probability which it is useful to idealize: (a) the long run idea, and (b) the idea of balanced causation. These two ideas rest on two mutually inconsistent philosophies. The first philosophy involves the supposition that the same set of antecedent events would not always produce the same result. The physical world is supposed to be so constituted that, if the trial of coin tossing were repeated indefinitely, *in exactly the same way each time*, heads and tails would happen equally often in the long run. Idealizing this notion, we have the familiar limit definition,

$$\lim_{n\to\infty} \frac{x}{n} = \frac{1}{2},$$

where x is the number of heads and n the number of trials. The second philosophy makes the requirement that any prescribed set of antecedent events is the determining cause of the result, and that the same result would always be produced from the same causes. We are, of course, not now concerned with the tenableness of either of these philosophies in man's world; we are merely assuming worlds in which the one or the other is true while constructing our definitions.

(*a*) The limit definition admits of the sum and product theorems, but unless carefully stated it leads to such inconsistencies as have been noticed recently by T. C. Fry[1] and W. A. Wilson.[2] First it is necessary to remark that the limit in question is not to be supposed uniform for all possible sequences. The major difficulty, however, is not a matter of uniformity; it has to do with the fact that the sequence we are considering does not exist *in the sense that* its various terms are not postulated or predetermined. It is not like the ordinary sequences of mathematics in which the law of formation of the terms is known, or at least contained implicitly in a set of known formulae. The inconsistencies mentioned

[1] *Probability and its Engineering Uses*, D. Van Nostrand Company (1928), pp. 89, 90.
[2] *Loc. cit.*, p. 578.

by Fry and Wilson are connected with this fact; also they are essentially the same though differing in form. The seeming paradox to which they allude may be put in a third form as follows: For any sequence of the sort demanded by our definition there would exist a number N (unknown) and a dependent number M such that, if the first M trials should be all heads, then the remaining $(N-M)$ trials would all be tails. It would appear, therefore, that, to an observer who might watch the first M trials, the probability of heads on the next trial would not be one-half, since that trial would be certain to result in tails. Such, however, is not the case, if one holds rigidly to the definition, for the sequence to which that next trial would belong, and of which it would constitute the first term, would also be one which would converge to one-half. There is nothing contradictory about tails being certain (in the popular sense) on that next trial and heads and tails being equally likely (in the sense of the definition).

These ideas may be expressed more succintly if one uses the language of point sets. There are an infinity of conceivable infinite sequences, of the sort demanded by our definition. Let a represent one such sequence, A the totality of the a's. Clearly A is a definite aggregate. Now choose $\epsilon>0$, and for each a denote by N the smallest n for which

$$\left|\frac{x}{n}-\frac{1}{2}\right|<\epsilon, \quad n \geqq N. \tag{1}$$

Next determine a subset A' of A such that, for each sequence of A', the following conditions hold: (*i*) There exists an M such that the first M trials are all heads and the next $(N-M)$ trials are all tails. Obviously M equals $N/2$, approximately. (*ii*) M is also such that, if the next $(N-M)$ trials were not all tails the inequality (1) would not hold. Clearly, A' is a definite subset. Now our postulate of convergence says that the sequence that interests us belongs to A. To obtain our so-called paradox one must postulate further that it also belongs to A'. This was done originally by the use of the phrase beginning, "*if* the first M trials should all be heads." This amounts to a partial particularization of the sequence, and it should not cause surprise that by such particularization one should also partially predetermine it. If one is to find a paradox in such partial particularization, one might as well go the limit and completely particularize it. A complete particularization would amount to a predetermination of every term. Let us do this, and let us again contemplate the immortal that Wilson describes, playing out the successive terms of the particular sequence we select. As Wilson states, N is unknown but fixed, now that the sequence is. In the same sense every term of this sequence is fixed, for we are now assuming that this immortal is actually playing out some particular a of A. This does not lead to a contradiction because, after any finite number of trials, he still stands at the threshold of a sequence of A. The probability of heads on his next throw is then one-half in the same sense as it was when he began.

Although this limit definition does seem to me to be free from real inconsistency, there are nevertheless valid objections to it. In fact, the idea it sets

out to idealize is not wholly satisfactory. One of the strongest objections is that it ignores what may happen in the short run. Suppose it were true that the coin tossing experiment would always lead to a peculiar subset B of A in which every sequence b had certain peculiar characteristics, such as long runs of heads or tails, then the condition of the definition would still be satisfied, but we might be quite unwilling to agree that heads and tails were under such conditions to be considered as equally probable. A part of this difficulty, but not all of it, is removed if one adds the requirement, which is in fact usually supposed implicit in the definition itself, that the conditions must remain true after any finite number of trials.

(*b*) Using the idea of balanced causation, and the deterministic type of philosophy, one may define heads and tails as being equally probable on the ensuing throw *if the causes which tend to produce the one are exactly equally balanced with those tending to produce the other in so far as we can measure them, provided that in truth they are not exactly balanced.* The last clause is needed in order to eliminate the cases where the coin will stay on edge. The major difficulty with this definition is that it is subjective, but this cannot be avoided; it is intrinsic in the idea of balanced causation. For, if we may imagine a god, all wise so far as past and present are concerned, and able to determine immediately the effect of all causes, to him probability could not exist in this sense. If the causes were not truly balanced he would know which way the coin would fall, and if they were exactly balanced it would stay on edge. Hence this idea can exist only for imperfect human intelligences; in other words it is of necessity subjective.

In the opinion of the author either of these definitions is logically tenable, but *not both at once.* They rest on two mutually exclusive philosophies, and much of the confusion that exists in discussions of probability is due to the fact that they are not kept distinct.

Turning now to the case where the number of elements is infinite, it would seem to be unnecessary for the statistician to consider it. Sampling from an infinite population is a convenient phrase commonly employed in statistics in the sense of sampling from a finite population with replacements. Further, there seems to be no reason why a frequency curve may not always be considered merely as a convenient approximation to a histogram, which in turn approximates a finite group. In fact, since in mathematics the word "infinite" applies to a process rather than to a fixed number, the physical concept of a bag containing an infinite number of balls does not exist. It is to be expected, therefore, that if one applies to the infinite realm phrases which have been given a meaning only for finite groups all sorts of paradoxes may appear, including the striking one suggested by Wilson.[1]

[1] *Loc. cit.*, p. 579.

A Simplification in the Numerical Computation of Multiple Integrals

By W. M. GOODHUE, Instructor in Electrical Engineering, Harvard University

In physics and in engineering there are many problems in which double integrals are to be evaluated, and in which the rapid determination of reasonably good approximate values is of prime importance. Suppose for example, that it is desired to compute the temperature due to a uniformly distributed source of heat, in a body whose shape and boundary condition permits the assumption that the problem is two-dimensional. This leads to the integral,

$$T_1 - T_0 = \frac{-\rho \operatorname{Div} J}{2\pi} \iint \left(\log \frac{r_1}{r_0} \right) dS, \tag{1}$$

where $T_1 - T_0$ is the temperature difference between two fixed points P_1 and P_0, r_1, r_0 are distances of dS from P_1 and P_0, J is the heat flow density vector, and ρ is the mean thermal resistivity of the material.

If this is to be computed by means of one of the usual approximation formulas, employing say n points for a single integration, then something like n^2 determinations of the integrand, depending on the shape of the region, and their sum, are required for the double integration.

The present paper suggests a transformation to make the integrand of the inner integral a function independent of the outer variable, so that the evaluation of this inner integral amounts to finding the area under a fixed curve, but between variable limits (the limits varying with the outer variable). Step by step integration naturally yields an *integral curve*, which is independent of the variable limits involved.

Expanding equation (1), and setting $dS = dx\,dy$,

$$T_1 - T_0 = \frac{-\rho \operatorname{Div} J}{2\pi} \iint \left(\log \frac{r_1}{y_1} - \log \frac{r_0}{y_0} + \log \frac{y_1}{y_0} \right) dx dy. \tag{2}$$

$$\text{Set } \frac{r_1}{y_1} = \sqrt{1 + \left(\frac{x_1}{y_1} \right)^2} = (1 + t_1^2)^{1/2} \tag{3}$$

where x_1 and y_1 are measured from point P_1 instead of the origin.

Equation (2) may now be further expanded:

$$T_1 - T_0 = \frac{-\rho \operatorname{Div} J}{2\pi} \left\{ \int y_1 dy \int_{t_1'}^{t_1''} \log (1 + t_1^2)^{1/2} dt_1 - \int y_0 dy \int_{t_0'}^{t_0''} \log (1 + t_0^2)^{1/2} dt_0 + \int (x'' - x') \log \frac{y_1}{y_0} dy \right\}.$$

Note that the inner integrals represent areas under the *same* curve, the dif-

ference being solely in the limits. The third term is merely a single integration. Allowing n points to each integration, the total number of points required is $3n$, assuming the inner integral curve available from previous problems, as it may be preserved and used indefinitely (otherwise n extra points).

A Note by O. D. Kellogg, Harvard University: Mr. Goodhue is here calling attention, by means of a specific example, to the reduction in labor of approximate computation which follows when it is possible so to change the variables in an iterated integral that the integrand becomes a product of functions, each of a single variable,—or even if it merely becomes the sum of a small number of such products.

Conditions for the possibility of such transformations are not difficult to set up, and, in the case of two variables, amount to a single differential equation for the two functions giving the transformation. As this is an underdetermined system, it looks as if the transformation were always possible. However, the equation is somewhat involved, and it would be of interest to determine broad classes of cases in which the reduction is possible by simple assigned methods. Such an investigation would also be of interest in the case of triple integrals, where the problem is no longer underdetermined.

On Equilateral Triangles

By J. R. MUSSELMAN, Western Reserve University

§1. In a recent number[1] of this Monthly, the following problem was proposed by Professor W. H. Echols: "*At the corners of any equilateral triangle* $A_1B_1C_1$ *let there be hinged three equilateral triangles* $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ *of any sizes or positions. Then will the mid-points of each of the sets of three segments* (A_2C_3, B_3C_2, A_3B_2), (B_1C_3, C_1B_2, B_3C_2), (A_2B_1, B_3A_1, A_3B_2), (A_1C_2, C_1A_3, A_2C_3) *be the corners of an equilateral triangle.*"

As there are a number of equilateral triangles connected simply with two given equilateral triangles and more connected with three given equilateral triangles, it seems worth while to call attention to them; and secondly, to point out in connection with Professor Echols's figure the existence of fifty-five equilateral triangles of which he mentions four. In conclusion I wish to state a theorem which is rather interesting and I believe entirely novel.

The most elegant method of proof follows from the use of a complex number as the coordinates of a point in the plane. Thus if the vertices of a triangle $P_1P_2P_3$ be given the complex numbers p_1, p_2, p_3 respectively, Professor Frank Morley[2] has shown that the condition that $P_1P_2P_3$ be a positively equilateral triangle i.e., that the verticesro tate into each other in a counter-clockwise direction, is

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[1] American Mathematical Monthly, vol. 39 (1932), p. 46. The notation in the problem has been changed to the above.

[2] Harkness and Morley, *Treatise on The Theory of Functions* (1893), p. 26.

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

or $p_1+\omega p_2+\omega^2 p_3=0$ where ω and ω^2 are the two complex cube roots of unity.

§2. Let $A_1A_2A_3$ and $B_1B_2B_3$ be the vertices of two positively equilateral triangles of any size or position whose coordinates in the plane are the complex numbers a_i, $b_i (i=1, 2, 3)$ respectively. These are subject to the conditions

$$a_1 + \omega a_2 + \omega^2 a_3 = 0 \tag{1}$$

$$b_1 + \omega b_2 + \omega^2 b_3 = 0. \tag{2}$$

If we divide the sum of (1) and (2) by the quantity 2 we obtain

$$(a_1 + b_1)/2 + \omega(a_2 + b_2)/2 + \omega^2(a_3 + b_3)/2 = 0, \tag{3}$$

whence we have the theorem *that the midpoints of A_1B_1, A_2B_2, A_3B_3 are the vertices of an equilateral triangle.* As both equations (1) and (2) are true when multiplied through by ω or ω^2, it is possible to combine them in two more ways, obtaining the result *that the midpoints* of A_1B_2, A_2B_3, A_3B_1; *also of* A_1B_3, A_2B_1, A_3B_2 *form the vertices of an equilateral triangle.* A simple way of obtaining them is in (3) to keep the a_i fixed and permute the b_i cyclically. Hence connected with two positively equilateral triangles of any size or position we find three other equilateral triangles.

§3. Let $C_1C_2C_3$ be a third positively equilateral triangle of any size or position whose vertices have been given the complex numbers c_1, c_2, c_3 respectively as coordinates. Then

$$c_1 + \omega c_2 + \omega^2 c_3 = 0. \tag{4}$$

If we combine (1), (2) and (4) by pairs in all possible ways as indicated in paragraph 2 we can obtain in all 9 equilateral triangles. Included among these are the three last triangles mentioned in the problem of Professor Echols.

If we add (1), (2) and (4) we obtain

$$(a_1 + b_1 + c_1) + \omega(a_2 + b_2 + c_2) + \omega^2(a_3 + b_3 + c_3) = 0. \tag{5}$$

Upon dividing this by the quantity 3 we have the theorem that *given three positively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ then the centroids of the three triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ themselves form a positively equilateral triangle.* Inasmuch as (1), (2) and (4) are true if the a_i, b_i, c_i $(i=1, 2, 3)$ are permuted cyclically, we see that (5) is only one of nine ways in which the points A_i, B_i, C_i can be arranged so that the centroids of the three triangles formed are themselves an equilateral triangle. To obtain all nine keep the a_i in (5) fixed and permute the b_i and c_i cyclically. Moreover we note that *the centroid of each of the nine equilateral triangles thus formed is the same point.*

§4. If the three positively equilateral triangles be now so placed in the plane that $A_1B_1C_1$ forms a positively equilateral triangle, we have the further requirement that

$$a_1 + \omega b_1 + \omega^2 c_1 = 0. \tag{6}$$

Multiply (2) by ω, (4) by ω^2 and add to their sum (1). Making use of (6) we obtain

$$(b_3 + c_2) + \omega(a_2 + c_3) + \omega^2(a_3 + b_2) = 0. \tag{7}$$

Dividing (7) by the quantity 2 we have thus proven the first statement in Professor Echols's problem that *the midpoints of B_3C_2, A_2C_3, A_3B_2 form an equilateral triangle.*

If we add (2), (4) and (6) we obtain

$$(b_1 + c_1 + a_1) + \omega(b_2 + c_2 + b_1) + \omega^2(b_3 + c_3 + c_1) = 0 \tag{8}$$

Upon dividing (8) by the quantity 3 we have the next theorem that *the centroids of the three triangles $B_1C_1A_1$, $B_2C_2B_1$, $B_3C_3C_1$ form a positively equilateral triangle.* To write down the nine equilateral triangles that can thus be formed keep the last letter in each term of (8), namely $a_1b_1c_1$, fixed and permute the other b_i and c_i cyclically. Thus for example, we obtain the theorem that the centroids of $B_3C_1A_1$, $B_1C_2B_1$,[1] $B_2C_3C_1$ form an equilateral triangle.

The existence of fifty-five equilateral triangles connected with the figure of three equilateral triangles, so placed in a plane that one vertex of each also forms an equilateral triangle, is now evident. If we take the equations (1), (2), (4) and (6) by pairs, from each pair by the method of paragraph 2 we obtain 3 equilateral triangles or 18 in all. If we take the equations by threes from each triad we obtain 9 equilateral triangles or 36 in all. Finally by combining all four equations we obtain one more, thus making a total of 55 equilateral triangles which can easily be constructed.

§5. If we place three positively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ so that $A_1B_1C_1$ is likewise a positively equilateral triangle we saw that the four conditions (1), (2), (4) and (6) could be combined to produce

$$(a_2 + c_3) + \omega(b_2 + a_3) + \omega^2(c_2 + b_3) = 0. \tag{7'}$$

Hence if in addition, we require $A_2B_2C_2$ to be a positively equilateral triangle, namely

$$a_2 + \omega b_2 + \omega^2 c_2 = 0,$$

then (7′) becomes $c_3+\omega a_3+\omega^2 b_3=0$, whence $A_3B_3C_3$ is likewise an equilateral triangle. Or in other words we have proven the following theorem: *given two positively equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$ of any size or position in the plane, if we construct the positively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$ and $C_1C_2C_3$ then $A_3B_3C_3$ itself is a positively equilateral triangle.*

In exactly similar way we can prove the theorem: *given two positively equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$ of any size or position in the plane, if we construct the negatively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$,* and *$C_1C_2C_3$ then $A_3B_3C_3$ is a positively equilateral triangle.*

[1] The centroid of $B_1C_2B_1$ is the point on the segment B_1C_2 one third of the distance B_1C_2 from B_1.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

A Treatise on Algebraic Plane Curves. By J. L. Coolidge. Oxford University Press, 1931. xxiv+513 pages. $10.00.

In the words of the author "the object of the volume before us" is "to give an account of the present status of the theory of algebraic plane curves, and their relations to various fields of geometry and analysis." A mere inspection of the table of contents would be almost enough to convince one that the author has done all that is possible with five hundred odd pages to accomplish his purpose.

The material is presented in four Books which deal respectively with the Elementary Theory (194 pp.), the Singular Points (53 pp.), Systems of Points on a Curve (127 pp.), and Systems of Curves (116 pp.). There is also appended an index of authors and a subject index.

The range of topics covered in these four books is extraordinarily wide. In the first book there are ten chapters devoted respectively to fundamental properties of polynomials, elementary properties of curves, real curves, real circuits of curves, elementary invariant theory, projective theory of singular points, Plücker's equations and Klein's equation, the genus, covariant curves, and metrical properties of curves. The point of view in this book is that of projective or Euclidean geometry.

The second book contains chapters on the reduction of singularities, developments in series, clustering singularities, and adjoint curves and Plücker's equations. Here Cremona transformations are used to reduce higher singularities to the ordinary type, and birational transformations to remove ordinary multiple points in favor of nodes.

The third book is mainly concerned with the algebraic geometry on a curve with excursions into the allied transcendental theory and a return at the close to the projective geometry of the rational curve. Its nine chapters discuss the general theory of linear series, abelian integrals, singular points of correspondences, moduli and limiting values, curves of special type, non-linear series of groups of points on a curve, higher theory of correspondences, parametric representation of the general curve (a sketch), and rational curves.

The eight chapters of the final book treat the postulation of linear systems by points, the transformation of linear systems, ternary apolarity, special curves in linear systems, non-linear systems of curves, the general Cremona transformations, and groups of Cremona transformations.

The encyclopedic character of the treatise, indicated by the chapter headings above, is strongly confirmed by an inspection of the topics considered in the

126 sections of the book. Many of these sections, after preliminary explanations, contain lists of theorems whose proofs, if reasonably direct, are merely indicated, or left entirely to the reader as exercises. In this way the scope of the work is much increased. In our opinion the volume not only will meet a long felt need on the part of students for a comprehensive survey in English of modern plane algebraic geometry, but also will be of real service to more advanced workers.

It would seem that on the whole the author has made a reasonably happy choice from the mass of material which was available for exposition. It is true that the excursions into the transcendental field are so sketchy that only expert readers can follow them, but doubtless little more could be attempted in one volume.

The arrangement of the content is not as logical as might be expected. One would naturally look for topics which belong essentially to projective geometry and the theory of invariants in the first book. But the study of the rational curve is postponed to Chap. IX of Book III, and that of ternary apolarity to Chap. III of Book IV. The references to the literature seem adequate for further orientation particularly in view of the existence of current surveys.

Some notations such as $\partial f/\partial 1$ may offend the conventional mind. At times the writer seems to prefer, for no obvious reason, the use of plain English to that of an appropriate technical term. Thus the classic separation of algebraic integrals into those of the first, second, and third "kinds" is always referred to as a separation into "sorts." The phrase—"a correspondence without value" (Wertigkeit)—will have an odd ring to one accustomed to the technical term "valence." Sic semper Coolidge?

So far as may be inferred from the preface this volume was prepared for the press without the aid of readers. As a result there are numerous slips of various kinds—misprints, mistakes in cross references, and faulty statements. For example, the theorems 13, 16, 21 (pp. 413–15) are either incorrect or badly stated. In the haste to reach results some errors in logic appear. Thus, on p. 206, in order to prove that the process of resolving a higher singularity eventually comes to an end, the theorem (p. 119) that the genus of an irreducible curve is not negative is used, even though in the intervening pages the theorem itself, and a method for calculating the genus, have not been derived for a curve with a higher singularity. Another example occurs on p. 487 where the following theorem is stated:

"A necessary and sufficient condition that a set of positive integers $n, r_1, r_2, \cdots, r_r$ which satisfy the relations,

$$3n - r_1 - r_2 - \cdots - r_r = 3, \quad n^2 - r_1^2 - r_2^2 - \cdots - r_r^2 = 1,$$

should correspond to an actual Cremona transformation is that there should exist positive integers s_j and such positive or zero integers α_{ij} that" the customary equations which condition these numbers should be satisfied.

The proof is merely a reiteration that the order n may be continually reduced by multiplication by a quadratic transformation until it becomes unity.

But if $n=5$, $r_1=r_2=3$, $r_3=r_4=\cdots=r_8=1$, a quadratic transformation at most reduces the case to $n=3$, $r_1=r_2=1$, $r_3=-1$, $r_4=r_5=\cdots=r_8=1$, and further reduction is impossible. It is easy in this particular case to verify that integers s_j, α_{ij} do not exist; but it would not be easy to make this verification in higher cases, and one must therefore assume provisionally that such integers do exist. There is of course little doubt that the theorem is true. It is likely that any valid proof will rest upon the existence of inequalities satisfied by the integers $n, r_1, \cdots, r_r$. The theorem is ascribed, in part at least, to S. Kantor but the reference given is incorrect.

Nevertheless these lapses do not seriously impair the value of the work as a whole. A very desirable feature is an exposition of the symbolic notation of Aronhold and Clebsch. The use of this notation should be as much a matter of course in projective and algebraic geometry as the use of vectors in metric geometry.

ARTHUR B. COBLE

The Macmillan Table Slide Rule. By J. P. Ballantine. New York, The Macmillan Company, 1931. $0.50.

This "Table Slide Rule" is a table of numbers printed on cardboard, in such a fashion that with the aid of cardboard slides it can be used somewhat as one does an ordinary slide rule, to read off directly the results, correct to three significant figures, of the common arithmetic operations of multiplication, division, root extraction, and so forth. There is the additional advantage that five figure accuracy is easily available from the table, though in this case not instantly. The idea is an ingenious one and deserves some attention. Since the table slide costs only fifty cents and seems to do the work, it is conceivable that it may go a considerable way toward displacing the slide rule in common use.

In comparison with the ordinary slide rule it is apparently at a disadvantage in at least two respects, namely portability and durability. The ordinary slide rule can be carried in the pocket and used in the two hands, but the table rule is not a pocket instrument and for comfortable operation requires two or three square feet of clear surface on which it can be laid out flat. Moreover the ordinary rule will outlast the present type of table rule many times over, but this difference is possibly compensated by the difference in initial cost.

An apparent advantage for the table rule is in the matter of setting. The ordinary slide rule must be set accurately to get good results. With the table rule there is a very comfortable margin of error in setting that produces no change at all in reading. This may well be a valuable contribution to the lessening of fatigue in extensive computation.

In an effort to get an impartial comparison, the present reviewer selected an intelligent member of his freshman class who had never seen or even heard of any kind of slide rule. This student was given both an ordinary slide rule and the table slide rule, with printed directions for the use of each. He was told to

learn how to multiply and divide and report forty-eight hours later for a proficiency test. It is an interesting fact that in the test, this student used the table slide not only more accurately but also more rapidly.

R. E. GILMAN

Bericht über Neuere Untersuchungen und Probleme aus der Theorie der Algebraischen Zahlkörper. By Helmut Hasse. Leipzig and Berlin, G. B. Teubner, 1930. Teil I: *Klassenkörpertheorie*, 134 pages. Teil II: *Reziprozitätsgesetz*, iv +204 pages.

During the last decade certain phases of the Theory of Algebraic Numbers have been materially advanced by the great work of Takagi and of Artin. Hasse has undertaken the task of presenting an exposition of the work of these mathematicians and of their followers.

A considerable portion of the first part of Hasse's report centers around Takagi's generalization and extensions of Hilbert's results on the class-field (Klassenkörper) of an Algebraic field. Takagi inculcated a new spirit in the treatment of relative abelian fields, by means of which that theory was greatly simplified and some previously unsolved problems settled with finality.

The *motif* of the second part of the report is Artin's reciprocity law, conjectured by Artin in 1923, and proved by him in 1926. Artin's reciprocity law may be heralded as the final reciprocity law which so many devotees of the Theory of Numbers have sought for a century. Its importance can hardly be overestimated and can be realized only by reading Hasse's report, from which it appears, among other matters, that the classical reciprocity laws of the Theory of numbers are consequences of Artin's reciprocity law.

A detailed bibliography and a long list of unsolved problems included in the report will be of invaluable assistance and serve as a stimulus to specialists in this fascinating but difficult branch of mathematics.

LOUIS WEISNER

The Queen of the Sciences. By E. T. Bell. Baltimore, The Williams and Wilkins Company, 1931. 138 pages. $1.00.

This little book belongs to "A Century of Progress Series" which, according to the announcement of the publishers, comprises "volumes by well-known scholars presenting the essential features of those fundamental sciences which are the foundation stones of modern industry." It is surprising how comprehensively Dr. Bell has presented the fundamental concepts of mathematics in a short non-technical treatise.

After an enlightening discussion of the object of mathematics in Chapter I, in which the fact is emphasized that creative mathematicians of all time have been inspired by "the art of mathematics rather than by any prospect of ultimate usefulness," the author continues in the next chapter with a description of mathematics and presents the famous definitions of Russell, Peirce, Klein and

Hilbert. The discussion leads naturally to the postulational method, algebra being treated by this method in Chapter III. The chapter presents also the parallel postulate and the characteristic features of the geometries of Lobachevsky and Riemann. Chapters following show that common algebra can be "realized" with the classes of rational, real, and complex numbers and present the notions of transformation, group, invariant, and matrix. To the "unruly domain" of arithmetic a chapter is devoted which contains, by way of interest, several simple classic, proved and unproved, theorems of the theory of numbers. Later chapters dwell on the notion of the infinite and on the power and present state of analysis. The concluding chapter is concerned with mathematical existence and with the modern critical attack on mathematical reasoning.

Dr. Bell, incidentally a successful novelist, has written in simple style, mathematically unconventional perhaps, but not flippant. The misspelling of the great name "Peirce" (page 18) and the statement of "88" for "68" (page 80) are obviously misprints. The reviewer, frankly enthusiastic, would like to recommend the book to everybody. It would be an unimaginative person who would not, from even a cursory reading, catch some of the glorious spirit of mathematics. The teacher who has not enjoyed the advantage of extensive modern graduate study will be instructed and oriented. The apprentice in the guild of mathematicians will be inspired. The university lecturer will put the book down feeling that he has reaffirmed his faith.

PAUL H. LINEHAN

Mathematics. By B. B. Low. London, Longmans, Green and Co., 1931. vii+443 pages.

The author states that he has avoided the title *Practical Mathematics* because of the difference of opinion as to what is practical, but his use of *Mathematics* and the purpose and scope of this volume are made clear by the following statement on page 1: "The scientific worker must understand certain parts of pure mathematics and must be able to apply his knowledge to obtain numerical results when these are required. He uses what has been called practical mathematics or what the author prefers to call *Mathematics*." This book should prove valuable to this scientific worker, either if he is confronted with some problem for whose solution his mathematical equipment is inadequate, or if he wishes to add to his knowledge of mathematics, within the author's meaning.

The subjects treated are elementary algebra, trigonometry, plane and solid analytic geometry, differential and integral calculus, derivation of empirical laws, calculus of finite differences, harmonic analysis, and differential equations. Very little previous mathematical training is assumed and there are numerous worked examples to illustrate each principle. Where it is impossible to make each subject complete in itself, the author attempts to make the way easier for the reader who seeks knowledge of some special topic by a fairly complete system of references to preceding chapters. The mathematically curious, however, will find need of other texts because, as is natural in this type of book,

the mathematical theory is developed no further than is necessary for the immediate end in view. A thoroughly sound list of such texts is found in Chapter I. There are numerous sets of not very difficult exercises, including many of scientific or technical interest.

L. T. MOORE

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1931–1932 should be submitted for publication not later than June 1, 1932.

CLUB TOPICS

1932 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

BY WALTER CROSBY EELLS, Stanford University, California

In continuation of previously published lists[1] of centennial dates in the history of mathematics, the following group of important 1932 centennial dates is presented.

B.C. 569.[2] Approximate date (according to Ball) of Pythagoras founder of the Pythagorean school of such great importance in the development of Greek mathematics.

A.D. 1532. Birth of Xylander, who translated the first six books of Euclid's *Elements* and works of other Greek mathematicians into German.

A.D. 1632. Birth of Christopher Wren, celebrated architect, who rectified the cycloid and determined its center of gravity.

A.D. 1632. Publication of Oughtred's *Circles of Proportion and the Horizontal Instrument* containing a description of his invention of the circular slide-rule.

A.D. 1632. Publication of Galileo's *Dialogo dei due massimi sistemi del mondo*, expounding the Copernican theory.

A.D. 1632. Death of Burgi, Swiss mathematical genius, who invented logarithms independently of Napier.

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[1] This Monthly, vol. 38 (1931) pp. 100–101 for a list of 1931 centennial events, and for references to previous volumes for corresponding lists from 1925 to 1930.

[2] This Monthly, vol. 38 (1931) p. 100, footnote, for justification of 569 B.C. as a centennial date for 1932.

A.D. 1632. Death of Girard, Dutch mathematician, author of treatise on trigonometry which, according to Ball, contains the first use of the abbreviations *sin*, *tan*, and *sec*.

A.D. 1732. Birth of Karsten, inventor of an interesting graphic representation of imaginary logarithms.

A.D. 1732. Birth of Rittenhouse, noted colonial mathematician and astronomer. "Without literary friends or society, and with but two or three books, he became, before he reached his four and twentieth year, the rival of two of the greatest mathematicians of Europe." (Cajori, *Mathematics in the United States*, Washington, 1890, pp. 38, 41, 42.)

A.D. 1832. Birth of Fiedler, who brought out German editions of Salmon's works on higher algebra and geometry.

A.D. 1832. Birth of Sylow, Norwegian mathematician, whose lectures materially extended the field of substitution groups. "Sylow's Theorem."

A.D. 1832. Publication of Jerrard's *Mathematical Researches* containing the reduction of the quintic equation to trinomial form.

A.D. 1832. Publication of Steiner's *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander*, one of the principal foundations upon which modern synthetic geometry rests.

A.D. 1832. Publication of Wolfgang Bolyai's *Tentamen juventutem in elementa mathaseos purae . . . introducendi*, the second volume of which, published the following year, included in an appendix his son's fundamental contribution on non-Euclidean Geometry.

A.D. 1832. Dirichlet gave a proof for Fermat's Theorem on the impossibility of $X^n+Y^n=Z^n$ in integers for $n=14$.

A.D. 1832. Death in a duel at the age of 20 of the youthful genius, Galois, to whom credit is due for important advances in group theory and the theory of algebraic equations of higher degree.

A.D. 1832. Death of Sadi Carnot whose essay *Réflexions sur la puissance motrice du feu*, according to Ball, "may be taken as initiating the modern theory of thermodynamics."

A.D. 1832. Suspension of publication of *The Mathematical Diary*, early American mathematical periodical.

A.D. 1832. Birth of Reverend C. L. Dodgson (Lewis Carroll) whose mathematical work was mainly in connection with Euclid and Symbolic Logic. (See "The Life and Letters of Lewis Carroll" by his nephew, S. Dodgson Collingwood, The Century Company, 1899.)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3552. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

Prove that

$$\begin{vmatrix} e_{11} & 0 & 0 & \cdots & 0 & A_1 \\ e_{21} & e_{22} & 0 & \cdots & 0 & A_2 \\ e_{31} & e_{32} & e_{33} & \cdots & 0 & A_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & e_{n2} & e_{n3} & \cdots & e_{n,n-1} & A_n \end{vmatrix}$$

$$= A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s},$$

where (1) $e_{ij}=1$ if j is a divisor of i and $e_{ij}=0$ if j is not a divisor of i; (2) p_i, $p_j \cdots$ are the distinct prime factors of n.

3553. *Proposed by A. A. Bennett, Brown University.*

Let c_r be defined for $r=1, 2, \cdots$, by

$$c_r = 0, \text{ if } r \text{ is not of the form } m(3m\pm 1)/2,$$
$$c_r = (-1)^m, \text{ if } r \text{ is of the form } m(3m\pm 1)/2.$$

Thus

r	1	2	3	4	5	6	7	8	9	10	11	$12\cdots$
c_r	-1	-1	0	0	$+1$	0	$+1$	0	0	0	0	$-1\cdots$

Show that the sum of the kth powers of the roots of

$$x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n = 0,$$

(for $k=1, 2, \cdots, n$), is s_k = the sum of the divisors of k, (independent of n). Thus,

k	1	2	3	4	5	6	7	8	9	10	11	$12\cdots$
s_k	1	3	4	7	6	12	8	15	13	18	12	$28\cdots$

3554. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Show by the processes of descriptive geometry how to determine the H- and V-projections of an elliptical cylinder (that is, a cylinder whose right section is an ellipse) whose H-base is a given ellipse and whose V-base is a circle.

3555. *Proposed by E. C. Kennedy, Texas College of Mines.*

If F_1 and F_2 are factors of the number 32,228,143 show that the inequality $3.2F_1 > F_2 > 3F_1$ cannot hold. Show that $2.2F_1 > F_2 > 2F_1$ and find the value of the two factors. F_1 and F_2 may be prime or composite but their product is equal to the original number. The problem is to be solved without making a single division or using any factor table, or table of primes.

3556. *Proposed by the late Artemas Martin, Washington, D. C.*

A hollow sphere, external and internal radii R and r, rolls down an inclined plane in time t; after the cavity is half filled with water it rolls down the same plane in time t'. Determine the specific gravity of the sphere.

See the *Annals of Mathematics*, vol. 8 (1894), p. 104.

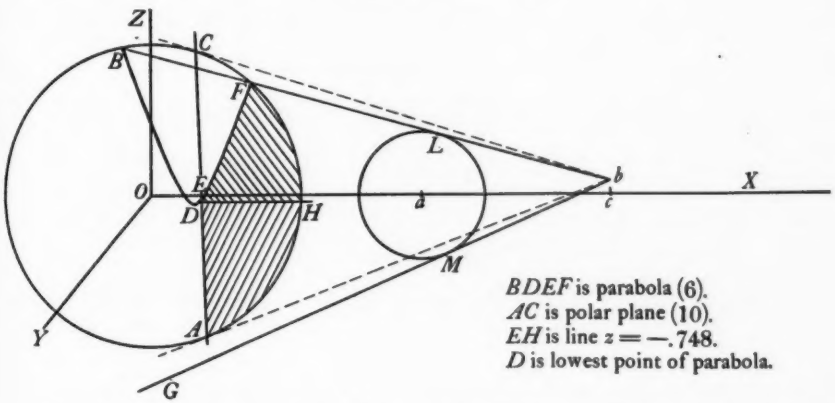
SOLUTIONS

353 (1914, 55; 1919, 312; 1931, 171). *Proposed by Richard P. Lochner.*

The center of a sphere, radius $R=5$ inches is $a=10$ inches above the surface of a sphere, radius $r=12\frac{1}{2}$ inches. There is a point of light at $b=1$ inch horizontally from a point $c=10$ inches vertically above the surface of the first sphere. What is the area of the shadow which the upper sphere casts on the lower one?

Solution by Wallace Smith, New River State College, Montgomery, West Virginia

Let us choose the centers of the spheres and point c on the X-axis and the position of the light above the X-axis in the direction of the Z-axis. Take the center of the sphere with radius $r=12\frac{1}{2}$ as the origin.



The above figure shows the shadow projected on the XZ-plane. The equation of the sphere at point a is

$$S_a \equiv (x - 45/2)^2 + y^2 + z^2 - 25 = 0. \tag{1}$$

The equation of the polar plane of point b with respect to S_a is

$$P_b \equiv 15(x - 45/2) + z - 25 = 0. \tag{2}$$

Hence a surface tangent to S_a at its intersection with P_b is given by

$$P_b^2 + kS_a = 0. \tag{3}$$

Now choose k so that the surface (3) passes through point b, then $k = -201$. Therefore,

$$P_b^2 - 201S_a = 0 \tag{4}$$

is a conical surface with vertex at b and tangent to S_a. The equation of the sphere at the origin is

$$S_0 \equiv x^2 + y^2 + z^2 - 625/4 = 0. \tag{5}$$

If we eliminate y between (4) and (5), we get the projection on the XZ-plane of the intersection of (5) with (4), which is

$$225x^2 + 30x(z - 61) + z^2 - 725z + 13075/4 = 0, \tag{6}$$

or

$$15x = 61 - z \pm 3(67)^{1/2}(z + \tfrac{3}{4})^{1/2}. \tag{6'}$$

This is the equation of a parabola with real points for $z \geqq -\frac{3}{4}$ and with a tangent parallel to the x-axis at $x = 247/60 = 4.1167$, $z = -\frac{3}{4}$. We now find the intersection of (6) with the great circle of (5) in the XZ-plane. Set $y = 0$ in (4) and we get

$$12x^2 + 15x(z - 61) - 25[4z^2 + (29/2)z - 1387/2] = 0, \tag{7}$$

or

$$24x = 15(61 - z) \pm 5(201)^{1/2}(z - 1), \tag{7'}$$

which is a pair of straight lines in the XZ-plane. If we set $y = 0$ in (5), we get

$$x^2 + z^2 - 625/4 = 0, \tag{8}$$

the trace of S_0 on the XZ-plane By eliminating x between (7') and (8), the positive radical furnishes the real values $z = 9.077$ and $z = 12.217$, while the negative radical gives only imaginary values. These are the values of z for the intersections of (6) with (8) also.

The polar plane of point b with respect to S_0 is

$$(75/2)x + z - 625/4 = 0. \tag{10}$$

Hence the illuminated part of S_0 when S_a is removed is the zone of S_0 toward b to right of (10). Therefore, the shaded part of S_0 is bounded by (4) and (10). We now find the intersections of (6) with (10), and (8) with (10) to be $x = 4.186$, $z = -.748$ and $x = 4.479$, $z = -11.670$ respectively. Now to determine the area we must consider the shaded portion as projected on the XZ-plane in two sec-

tions: first, take that portion between the parabola (6′), the great circle (8), the point at $z=9.077$, and the line $z=-.748$; second, take that portion between the polar plane (10), the great circle (8), the line $z=-.748$, and the point at $z=-11.670$. Hence the area of the shaded portion of S_0 is given by the sum of the two integrals:

$$A = 25\int_{-.748}^{9.077}\int_{f_1(z)}^{f_3(z)} y^{-1}dxdz + 25\int_{-11.670}^{-.748}\int_{f_2(z)}^{f_3(z)} y^{-1}dxdz,$$

where

$$y = (625/4 - x^2 - z^2)^{1/2}, \quad f_1(z) = [61 - z + 3(67)^{1/2}(z + \tfrac{3}{4})^{1/2}]/15$$

$$f_2(z) = 25/6 - 2z/75, \qquad f_3(z) = (625/4 - z^2)^{1/2}.$$

Note: Professor Dunkel suggested the simple method of writing the equations (4) and (10) which lead to the finding of the desired limits.

3496. [1931, 340] *Proposed by H. E. Stelson, Kent State College.*

In the triangle ABC, a normal is drawn to the circumcircle cutting the lines BC and AC at X and Y respectively. Find the locus of intersection of AX and BY.

Solution by A. Pelletier, Montreal, Canada

Let O, the center of the circumcircle, be taken as the origin so that the coordinates of A, B, C are $(-a, b)$, (a, b), (c, d), respectively, and $y=mx$ is the equation of YOX. The coordinates of X are then $(b-ka)\ (m-k)^{-1}$, $m(b-ka)(m-k)^{-1}$, where $k=(d-b)\ (c-a)^{-1}$. The equation of AX is then

$$(y - b)(b + ma - 2ka) = k(b - ma)(x + a). \tag{1}$$

The equation of BY is obtained by replacing a by $-a$ and k by $k'=(d-b)(c+a)^{-1}$.

Eliminating m from these two equations, we get for the equation of the locus of the intersection P of AX and BY

$$(y - b)^2(ka - k'a - b) - 2kk'a^2(y - b) + kk'b(x^2 - a^2) = 0. \tag{2}$$

An examination of (2) shows that, if O lies within ABC, the locus is an hyperbola; if O lies outside ABC, it is an ellipse; if O lies on AB, it is a straight line through C parallel to AB.

A Note by Otto Dunkel: If O is any fixed point in the plane of the given triangle ABC, the lines OXY cut BC and AC in two projective ranges of points X and Y. Hence P, the intersection of AX and BY, describes a conic through A and B with tangents OA and OB. The vertex C is also on the conic, since at C X, Y, and P coincide. If CO cuts AB in N, and if N' is taken on AB so that

N', A, N, B are harmonic, then $N'C$ is the tangent at C. The three tangents at A, B, C intersect in two other points O_a and O_b, and any one of the points O, O_a, O_b may be used to generate the conic with a suitable pair of vertices of ABC.

If O lies anywhere within angle B, or its vertical angle, i.e. within the infinite angular region containing the area ABC, it is easily seen that OXY may be drawn so that P lies within the triangle ABC. Hence in such cases the conic must be an hyperbola. A similar remark applies to the angular region at A. There are other regions for O which determine an hyperbola. If AX and BY are parallel, P is at infinity. Consider then the equal pencils AX, BY, where corresponding rays are parallel. Then the straight line XY envelopes a conic tangent to AC and BC. When X is at C, Y is at infinity on AC; and hence this conic is tangent to AC at infinity. The conic is therefore an hyperbola with AC and BC as asymptotes. If $A'B'$ is the straight line symmetric to BA with respect to C, it is clear that $A'B'$ is tangent to the hyperbola. By symmetry it follows that AB is also a tangent. Hence if O lies within this hyperbola, the locus of P is an ellipse; if O lies outside, the locus of P is an hyperbola; if O lies on the hyperbola, the locus of P is a parabola. If O lies on any side of ABC, the locus of P degenerates.

In the special case in which O is the center of the circumcircle, the only position of O that needs additional consideration is one for which O and C are separated by AB. In this case it is obvious that in the triangle OO_aO_b the points A, B, C lie on and within the segments of its sides, and we surely get an ellipse as the locus of P. This shows that in this case the center of the circumcircle lies within the hyperbola.

Also solved by D. C. Duncan, F. Underwood, and F. L. Wilmer.

3497. [1931, 340] *Proposed by J. Rosenbaum, Milford, Conn.*

Denoting the coefficients of the expansion of $(x+y)^n$, n being a positive integer, by $a_0, a_1, \cdots, a_n$; find the sum of $a_0, a_3, a_6, \cdots, a_{3k}$, where $(n-2) \leqq 3k \leqq n$.

Solution by Morgan Ward, California Institute

Suppose that

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{1}$$

is any polynomial in z of degree n, and that it is desired to evaluate the sum

$$S = a_s + a_{s+r} + a_{s+2r} + \cdots + a_{s+kr}$$

where

$$s + kr \leqq n < s + (k+1)r, \quad s < r.$$

Let ω be a primitive rth root of unity, so that

$$\omega^t + \omega^{2t} + \omega^{3t} + \cdots + \omega^{rt} = 0, \quad t \not\equiv 0 \bmod r$$
$$= r, \quad t \equiv 0 \bmod r. \tag{2}$$

Multiply (1) by z^{r-s}, set $z=\omega^t$, and then sum from $t=1$ to $t=r$. On simplifying the resulting expression by the use of (2), we obtain the general formula

$$S = \frac{1}{r}\left\{f(\omega)/\omega^s + f(\omega^2)/\omega^{2s} + \cdots + f(\omega^r)/\omega^{rs}\right\}. \tag{3}$$

Now let us assume that $s=0$, and that $f(z)=(1+z)^n$. Then on taking $\omega=e^{2\pi i/r}$, we have for any positive integer t,

$$1 + \omega^t = 2e^{\pi it/r}\cos(\pi t/r), \quad 1 + \omega^{r-t} = 2e^{-\pi it/r}\cos(\pi t/r)$$

so that

$$f(\omega^t) + f(\omega^{r-t}) = 2^{n+1}\cos\frac{n\pi t}{r}\left(\cos\frac{\pi t}{r}\right)^n. \tag{4}$$

On simplifying the right side of (3) by the use of (4), we find after a few reductions that

$$S = \frac{2^{n+1}}{r}\left\{\tfrac{1}{2} + \cos\frac{n\pi}{r}\left(\cos\frac{\pi}{r}\right)^n + \cos\frac{2n\pi}{r}\left(\cos\frac{2\pi}{r}\right)^n + \cdots\right.$$
$$\left. + \cos\frac{kn\pi}{r}\left(\cos\frac{k\pi}{r}\right)^n\right\}.$$

k here is the greatest integer in $(r-1)/2$.

In the case at hand, $r=3$, so that $k=1$ and

$$S = \frac{2^{n+1}}{3}\left\{\tfrac{1}{2} + \cos\frac{n\pi}{3}\left(\cos\frac{\pi}{3}\right)^n\right\}$$
$$= 2^n/3 + (-1)^n 2/3, \quad n \equiv 0 \bmod 3$$
$$= 2^n/3 + (-1)^{n+1}/3, \quad n \not\equiv 0 \bmod 3.$$

Also solved by R. P. Agnew, Frank Ayres, H. D. Grossman, H. L. Olson, W. V. Parker, A. Pelletier, and F. Underwood.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The Sixteenth Summer Meeting of the Mathematical Association of America will be held at the University of California at Los Angeles on Monday and Tuesday, August 29–30, 1932, in conjunction with the summer meeting and colloquium of the American Mathematical Society. The program of the Association sessions will be sent to members the middle of July. The colloquium lectures will be given by Professor J. F. Ritt of Columbia University on "Differential equations from the algebraic standpoint." By invitation Professor D. N. Lehmer will address the Society on "The continued fraction representing cubic and higher irrationalities," and Professor Tibor Radó on "Recent work in the problem of Plateau." The joint dinner will occur Wednesday evening. On Thursday afternoon and evening an excursion will be held to visit the California Institute of Technology and the Mount Wilson Observatory at Pasadena. Opportunity will also be given to visit the University of Southern California, Occidental College, Pomona College, and other institutions of this region. Arrangements will be made for tennis and golf.

Accommodations for the mathematicians will be furnished in Mira Hershey Hall, a new women's dormitory of the University of California at Los Angeles. The charge for a single room will be $1.50 per person per day, and $1.00 per person per day when the room is occupied by two, provided the room is engaged for a minimum of three days. Suitable arrangements can be made for families with children. Meals will be served at Mira Hershey Hall at the rate of 50c for breakfast, 75c for luncheon, and 75c for dinner.

The railroads are planning the usual summer rates to California and it will be possible to obtain a round-trip rate whether one uses a northern route or not, thus making it possible to go or come from the east by a southern or central route and to make the other trip through the Northern Rockies or Canada, including a visit to the Pacific Northwest, without incurring any extra railroad charges. This makes it possible for those planning to attend the meeting at Los Angeles to arrange a vacation in any part of the Rockies or along any portion of the Pacific Coast.

The Olympic Games are to be held at Los Angeles during July and August, most of the main track events being in August. Those attending these games should make hotel reservations well in advance.

The French government has conferred the title "Officer de l'Academie" on Professor E. R. Hedrick, of the University of California at Los Angeles, "for services rendered to the cause of culture and of science."

Dr. Irving Langmuir, of the General Electric Company, has been awarded the prize of the Popular Science Monthly for notable scientific achievement.

Sir James Hopwood Jeans has been elected an honorary member of the Washington Academy of Sciences.

Professors Wilhelm Blaschke, of the University of Hamburg, in 1931 Visiting Lecturer of the American Mathematical Society, lectured in February, 1932, at the Universities of Annamalai and Calcutta, and will lecture in April at the Universities of Tokyo and Sendai, in May at Stanford University, and during the summer session at the University of Chicago. His subjects will be differential geometry, topology, and continuous groups.

Professor Albert Einstein has been appointed Rouse Ball lecturer at the University of Cambridge for the year 1931–32. Professor Einstein has been in residence at the California Institute of Technology and the Mount Wilson Observatory during January and February, 1932.

Dr. Willem de Sitter, director of the Astronomical Observatory of the University of Leiden, delivered a series of lectures in January, 1932, at the University of California, on *The astronomical aspects of the theory of relativity*, and *The system of astronomical constants*. He has also lectured recently at several other American universities.

Dr. W. A. Shewhart, of the Bell Telephone Laboratories, will lecture at the University of London in the spring of 1932 on the role of statistical method in industrial standardization, and will hold a series of conferences on theoretical and applied statistics in Great Britain and on the Continent. Dr. Shewhart has recently been elected a fellow of the Royal Statistical Society of England.

The summer meeting of the American Association for the Advancement of Science will be held at Syracuse N. Y., June 20 to 25. Section A of the Association will meet on Tuesday, June 21. The program of this section will be as follows: Professor H. M. Gehman, University of Buffalo, will speak on "Homeomorphic Geometry of the Projective Plane" at 10:00 A.M. Professor W. A. Hurwitz, Cornell University, on "Logical Foundations for Groups and Fields" at 11:15 A.M. Professor J. Shohat, University of Pennsylvania, on "Interpolation" at 2:00 P.M. The mathematicians attending this meeting will have lunch together, and after the afternoon session there will be a picnic at a nearby lake.

The announcement has been received of the death of Professor Heinrich Wieleitner, Oberstudiendirektor am neuen Realgymnasium und Honorarprofessor an der Universität, München, on December 27, 1932. Professor Wieleitner was well known to all students in the history of mathematics. He was a serious scholar and a clear and accurate writer. His loss will be felt in this country as well as in Munich, where he worked for so many years.

The thirteenth annual meeting of the National Council of Teachers of Mathematics was held on February 19–20, 1932, at Washington, D. C. The following papers were presented:

1. "Curriculum adjustments" by W. C. Myers, Washington, D. C.
2. "A comparative study of the teaching of mathematics in the United States and Germany" by Professor W. D. Reeve, New York, N. Y.
3. "Improving America's mathematics" by H. C. Barber, Exeter, N. H.
4. "What do we owe to the brighter pupil?" by Beulah I. Shoesmith, Chicago, Ill.
5. "Regents' and College Entrance Board examinations in mathematics" by Professor W. S. Schlauch, New York, N. Y.
6. "What mathematics means to the world" by Professor E. R. Hedrick, Los Angeles, Calif.

All of these papers will appear in forthcoming issues of The Mathematics Teacher.

Three new standing committees were established, one on individual differences, one on geometry, and a third on policy, which is to formulate a body of first principles that should govern the teaching of secondary mathematics. The committee on cooperation with official examining boards was continued for another year, as was the committee on handbook in mathematics.

The annual election resulted as follows: President, William Betz, Specialist in mathematics, Rochester, N. Y.; Second vice-President, Mary A. Potter, Supervisor of mathematics, Racine, Wis.; Directors, Mrs. Elsie P. Johnson, High School, Oak Park, Ill.; J. P. Everett, State Teachers College, Kalamazoo, Mich.; and Raleigh Schorling, University of Michigan, Ann Arbor, Mich. E. W. Schreiber, State Teachers College, Macomb, Ill., continues as Secretary-Treasurer.

The National Council of Teachers of Mathematics, which had a very modest beginning in Cleveland, Ohio, February 1920, when approximately seventy-five charter members created the organization, has now grown to a membership of nearly six thousand, representing every state in the union and many foreign countries.

The following courses in mathematics are announced for the summer 1932:

University of Southern California, first term, June 17 to July 29. In addition to the usual elementary work, the following advanced courses are offered: By Professor L. D. Ames: Selected topics—Algebra; Theory of functions of a complex variable. By Professor L. E. Gurney: Mathematical astronomy. By Associate Professor D. V. Steed: Vector analysis; Riemannian geometry. Second term, July 29 to September 2. By Professor Ames: Infinite processes; Modern higher algebra. By Professor Gurney: Differential equations; Mathematical astronomy.

The Mathematical Association of America

Invites all persons interested in the promotion of mathematics in America, especially in the collegiate field, to become members of this Association, and it urges you who are now members to extend this invitation personally and most cordially to all non-members within your circle of convergence.

The Association is living well within its budget, but some of its revenues are affected by the prevailing depression. Hence this added reason for a constantly increasing membership. Also note the statement of the Secretary on page 134 of the March, 1932 "Monthly."

It was a deep conviction of the late lamented Professor Young that the Association should extend its influence far more widely not only among college students but also among intelligent people in general, to the end that the rôle which mathematics plays in the world might be more widely understood and appreciated. This conviction was strongly set forth in his retiring presidential address printed in the January, 1932 "Monthly," pages 1-15. This address will bear reading and rereading.

The Editorial Board has some plans on foot which may be a beginning toward a partial realization of some of Professor Young's ideals, at least so far as these may be realized in the present Departments of the "Monthly."

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY, Lexington, May.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA, River Falls, Wis., May 7.
MISSOURI.
NEBRASKA, Omaha, May, 6-7.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

John Wesley Young, 1879-1932

JOHN WESLEY YOUNG

Appreciation by President Hopkins of Dartmouth College

Perhaps the informal and companionable attributes of Professor Young's character can best be illustrated by the remark of a mutual friend the day of Professor Young's death. This man, bespeaking his sense of loss, said: "I never knew until this morning that Cy's name was John Wesley." This was actually true, I believe, of many another. Certainly over nearly two decades of acquaintanceship and friendship I never knew any official or unofficial occasion when formality had to be summoned to substantiate the position of either one of us in our mutual relationships.

The quality of his scholarship was beyond question, of course, but he was a scholar who liked to teach. His interest, moreover, was as keen in individual students as in the subject. He loved and respected both, I think. As an official associate the breadth of his interest in affairs not only of the curriculum but of general academic policy made him invaluable in discussion and deliberation concerning College affairs. He was wise in counsel, loyal in friendship, highly qualified for work of distinction in his chosen field, and indefatigable in his eagerness to know about and understand the world in which he lived. We who knew him well value greatly the privileges which have been ours in association with him, and in intimate thought we cherish his memory.

Appreciation by a Student Friend

(From The Dartmouth of February 20, 1932)

VALE: CY YOUNG

A week ago: your hand, in my hand resting—
The shy, queer smile that never went away.
A week ago: your jest, to crown my jesting—
Sunlight at nightfall on a gloomy day.

Long I shall cherish that last careless meeting,
That picture, stamped unfading on the brain:
Symbol of you, who offered life a greeting
Of banter, for its surly gift of pain.

Great heart, grown quiet, merely to have seen you
Wear, as an honored garment, such great ruth,
Made us your debtors. Death cannot demean you
Who taught us manhood in our stumbling youth.

A. L.